Reflected BSDE's , PDE's and Variational Inequalities

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Abstract: We discuss a class of semilinear PDE's with obstacle, of the form

$$(\partial_t + L)u + f(t, x, u, \sigma^* \nabla u) + \mu = 0, \quad u \geq h, u_T = g$$

where h is the obstacle. The solution of such an equation (in variational sense) is a couple (u, μ) where $u \in L^2([0, T]; H^1)$ and μ is a positive Radon measure concentrated on $\{u = h\}$.

We prove that this equation has a unique solution and u is the maximal solution of the corresponding variational inequality. The probabilistic interpretation (Feynman-Kac formula) is given by means of Reflected Backward Stochastic Differential Equations. We give a new construction of solutions of such equations using a maximum principle. This perimts to consider obstacles with jumps.

Key-words: Reflected Backward Stochastic Differential Equations, Variational Inequalities, Stochastic Flows

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EDSR réfléchis, EDP et Inégalité Variationnelles

Résumé : On discute une classe d'EDP semilinéaires avec obstacle de la forme

$$(\partial_t + L)u + f(t, x, u, \sigma^* \nabla u) + \mu = 0, \quad u \geq h, u_T = g$$

où h est l'obstacle. La solution d'une telle équation (en sens varionnel) est un couple (u, μ), avec $u \in L^2([0,T]; H^1)$ et μ mesure de Radon positive concentrée sur $\{u = h\}$.

On prouve l'existence et l'unicité de la solution et on prouve ainsi que u est la solution minimale de l'inégalité variationnelle correspondante. L'interprétation probabiliste (formule de Fayneman-Kac) est donnée par le biais des Equations Différentielles Stochastiques Rétrogrades Réfléchies. Nous présentons une nouvelle construction de la solution d'une telle équation en employant un principe de maximum. Ceci permet de traiter des obstacles avec des sauts.

Mots-clés : Equations Différentielles Stochastiques Rétrogrades Réfléchies, Inégalitées Variationnelles, Flots Stochastiques

1 Introduction

Backward Stochastic Differential Equations (in short BSDE's) were first introduced by E. Pardoux and S. Peng in [P.P.1] and have many applications in mathematical finance, stochastic control and stochastic games. Their initial motivation was to give a probabilistic interpretation for the viscosity solutions of semi linear PDE's (a generalization of Feynman-Kac formula - see [P.P.2]). Recently, G. Barles and E. Lesigne [Ba.L.], and afterwards V. Bally and A. Matoussi [B.M] studied the relation between BSDE's and solutions of semi linear PDE's in Sobolev spaces. This is also the point of view in this paper.

In a recent paper, N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M.C. Quenez [El-K.K.P.P.Q] introduced the Reflected BSDE's (in short RBSDE's): these are BSDE's in which an increasing process K is introduced in order to force the solution Y of the BSDE to remain larger as a process S which represents an obstacle. They proved that the solution of a reflected BSDE is the value function of an optimal stopping problem and provide a probabilistic interpretation for the viscosity solution of an obstacle problem associated with a non linear parabolic PDE.

Obstacle problems for a semi linear PDE's and their connection with optimal stopping and control problems have already been studied by A. Bensoussan and J.L. Lions [Be.L]. They model such problems by means of variational inequalities.

Our paper contributes to this topic in two directions. First of all we give (see Section 2) a new construction of the solution of RBSDE's using a maximum principle. This appears as an alternative to the classical penalization method used both in [El-K.K.P.P.Q] and in [Be.L]. It is maybe interesting to note that the penalization method produces the solution of a RBSDE as an increasing limit of a sequence of solutions of standard (non - reflected) BSDE's since our method employs a decreasing sequence of such solutions. So we obtain a lower and an upper bound and this may be interesting from a numerical point of view because the speed of convergence of the approximation scheme based on penalization is not aviable. We also menation that the construcion presented in this paper permits to prove existance and uniqueness for RBSDE's with an obstacle which may have jumps, while the obstacle is continuous in [El-K.K.P.P.Q].

The second point (see Section 3) is to establish the relation betwen RBSDE's and variational inequalities. In fact we give a new variational formulation of the obstacle problem which permits to prove uniqueness of the solution. Recall that for variational inequalities one has a minimal solution but uniqueness does not hold except if the obstacle is very smooth (it belongs to the domain of $\partial_t + L$ where L is the second order differential operator in the PDE). In our frame the obstacle is assumed to be just continuous, which is the case in many interesting applications. The solution of the obstacle problem is, in our formulation, a couple (u, μ) where u is in fact the solution of the variational inequality and μ is a positive measure which acts only when u hits the obstacle. In some sense μ represents the quantity which permits to pass from the inequality to equality (the missing term). Using probabilistic techniques (based on RBSDE's) we prove that there exists a unique couple (u, μ) which solves our problem. Reacall also that the solution of a RBSDE is a triplet (Y, Z, K) and the relation with u is given by $u(s, X_s) = Y_s$ and $\sigma\nabla u(s, X_s) = Z_s$. So we have a probabilistic interpretation of u and of ∇u. In our formulation we also obtain the relation betwen the increasing process K and the measure μ (see Theorem 7,b)ii)).

The present paper is a shortened version of the work presented in the preprint [B.C.F] in which we give some further developpments concerning the relation betewen RBSDE's and optimal stopping and control problems on one hand and we discuss RBSDE's with two barriers and the connection with obstacle problems on the other hand. These topics have already been discussed in [El-K.P.Q] and [El-K.Q] (for optimal stopping) and [C.K] and [H.L] for RBSDE's with two barriers and the relation with game problems. Since (except technicalities) the ideeas are essentially the same as in the case of a single barrier, we leave these topics out and send the interested reader to the preprint.

Acknowledgment. We thank Nicole El-Karoui for usefull discutions, especially concerning the maximum principle presented in Proposition 3 of the paper.

2 Construction of a solution for RBSDE's

2.1 Regular obstacle

On a probability space $\{\Omega, \mathcal{F}, P\}$ we consider a Brownian motion $B_t = (B_t^1, ..., B_t^d), 0 \leq t \leq T$, and denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the standard filtration generated by B, i.e. $\mathcal{F}_t$ is the completion of $\sigma(B_s; s \leq t)$ with respect to $(\mathcal{F}, P)$. We denote $\diamondsuit$ H^2 the space of predictable processes $\phi : [0,T] \times \Omega \to R$ which are square integrable, i.e. $E\int_0^T \phi^2(t,\omega)dt < \infty$.

$\diamondsuit$ $\mathcal{S}$ the space of continuous adapted processe ϕ such that $E\sup_{t \leq T} |\phi_t|^2 < \infty$.

The following three objects are given (H_1) The driver $f : [0,T] \times \Omega \times R \times R^d \to R$ which is a function such that:

$$
(1) \quad \begin{array}{lll}
i) & f(\circ, y, z) & \in H^2 \text{ for every } y, z, \\
ii) & |f(t,\omega,y,z)| & \leq K(1 + |y| + |z|), \\
iii) & |f(t,w,y,z) - f(t,w,y',z')| & \leq C(|y - y'| + |z - z'|)
\end{array}
$$

(H_2) The obstacle

$$
S_t = \int_0^t (V_s, dB_s) + \int_0^t U_s ds + A_t
$$

where $V = (V^1, ..., V^d)$ and U are predictible processes such that $E\int_0^T |V_t|^2 dt + E\int_0^T |U_t| dt < \infty$ and A is an adapted, cadlag process which has finite variation and such that $E\sup_{t \leq T} |A_t| < \infty$. We assume that dA_t is a positive measure which is singular with respect to the Lebesgue measure (so the above decomposition is unique).

We will also consider the following supplementary hypothesis: (H_3) The terminal value : $\xi \in L^2(\Omega, F_T, P)$. We assume that $\xi \geq S_T$.

Remark 1 *The process A, and therefor S, may have positive jumps, so the hypothesis here are slightly more general then in [El-K.K.P.P.Q] where S is continuous.*

We consider now the RBSDE

$$
\begin{aligned}
(2) \qquad & \mathbf{Eq}(\xi, \mathbf{f}, \mathbf{S}) \\
Y_t \; = \; & \xi + \int_t^T \left\{ f(s,\omega,Y_s,Z_s) + \alpha_s 1_{\{Y_s=S_s\}} (f(s,\omega,S_s,V_s)+U_s)^- \right\} ds \\
& - \int_t^T Z_s dB_s
\end{aligned}
$$

and we look for a solution $Y_t \geq S_t$.

More precisely, a solution of the RBSDE $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S})$ is a triple $(Y_t, Z_t, \alpha_t)_{0 \leq t \leq T} \in (H^2)^{d+2}$ which verifies $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S})$ and such that $Y_t \geq S_t, \forall 0 \leq t \leq T, a.s.$

Remark 2 *We may also put $\alpha_s 1_{\{Y_s=S_s\}}(f(s,\omega,Y_s,Z_s) + U_s)^-$ instead of $\alpha_s 1_{\{Y_s=S_s\}}(f(s,\omega,S_s,V_s) + U_s)^-$ in the above equation. In fact nothing changes: since we are on the set $Y_s = S_s$, it does not metter if we put S or Y. On the other hand, since $Y - S$ is a semimartingale with the martingale part given by $(Z_s - V_s)dB_s$, it is known that $1_{\{Y_s - S_s = 0\}}(Z_s - V_s)^2 ds = 0$, so that $1_{\{Y_s - S_s = 0\}}(Z_s - V_s)^2 = 0, ds$ almost surelly. This permits to replace V by Z.*

The main result in this section is the following.

Theorem 1 *Assume $(H_1), (H_2)$ and (H_3). Then there exists a unique $(Y_t, Z_t, \alpha_t)_{0 \leq t \leq T} \in (H^2)^{d+2}$ solution of the RBSDE $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S})$. Furthermore, Y is a continuous process, $E(\sup_{t \leq T} |Y_t|^2) < \infty$ and $0 \leq \alpha \leq 1, ds \times dP, a.s.$*

Remark 3 *The uniqueness property holds in the following sense: if $(Y_t, Z_t, \alpha_t)_{0 \leq t \leq T}$, $(Y'_t, Z'_t, \alpha'_t)_{0 \leq t \leq T} \in (H^2)^{d+2}$ are two solutions, then $Y_t = Y'_t, \forall 0 \leq t \leq T, a.s,$ $\quad Z_t = Z'_t, dt \times dP, a.s.$ and $\alpha_s = \alpha'_s, 1_{\{Y_s=S_s\}}(f(t,\omega,Y_s,Z_s) + U_s)^- ds \times dP, a.s.$*

Remark 4 *In the hypothesis (H_2) we ask dA_t to be a positive measure and theis excludes the case of negative jumps - which may be interesting for solving impulsional control problemes for example. In fact this is a quait superficial restriction, and once we can solve problems with positive jumps we may treat problems with negative jumps as well, just using a tranzlation procedure. Write $A = A^+ - A^-$ and suppose that existance and uniqueness have already been proved if $A^- = 0$. Then we define $\widetilde{\xi} = \xi + A^-_T, \widetilde{S} = S + A^-$, and $\widetilde{f}(s,\omega,y,z) = f(s,\omega,y - A^-_s(\omega), z)$ and we note that a triplet (Y, Z, K) verify $Eq(\xi, f, S)$ if and only if $\widetilde{Y} =: Y + A^-, \widetilde{Z} = Z$ and $\widetilde{K} = K - A^-$ solves $Eq(\widetilde{\xi}, \widetilde{f}, \widetilde{S})$. So existance and uniqueness for the equation $Eq(\xi, f, S)$ follows from the existance and uniqueness for the equation $Eq(\widetilde{\xi}, \widetilde{f}, \widetilde{S})$, for which $A^- = 0$.*

The main tool in the proof of Theorem 1 is the following maximum principle.

Proposition 1 *Assume* $(H_1), (H_2)$ *and* (H_3) *and assume also that for* $P(d\omega)$ *almost every* ω

$$(H_4) \quad (f(s,\omega,S_s,V_s)+U_s)ds+dA_s \geq 0.$$

Then the solution (Y,Z) *of the standard BSDE*

$$\mathbf{Eq}(\xi,\mathbf{f}) \quad Y_t=\xi+\int_t^T f(s,\omega,Y_s,Z_s)ds-\int_t^T (Z_s,dB_s)$$

satisfies $Y_t \geq S_t$. *In particular* $(Y,Z,0)$ *solves* $\mathbf{E}q(\xi,\mathbf{f},\mathbf{S})$.

Remark 5 *Since* dA_t *is singular with respect to the Lebesgue measure the above condition is equivalent with* $dA_t \geq 0$ *(with already appears in our hypothesis* (H_2) *and* $f(t,\omega,S_s,V_s)+U_s \geq 0, a.s.$

Poof. We just adapt the argument used for proving the comparisontheorem for standard $BSDE's$ (if $A=0$ our proposition is animmediate consequence of the comparison theorem). We define $y_t=Y_t-S_t, z_t=Z_t-V_t, \theta_t=f(t,\omega,S_t,V_t)+U_t$ and x

$$\begin{aligned}
\alpha_t &= \frac{f(t,\omega,Y_t,Z_t)-f(t,\omega,S_t,V_t)}{Y_t-S_t} \quad \text{if} \quad Y_t \neq S_t \\
&= 0 \quad \text{if} \quad Y_t=S_t, \\
\Gamma_{t,T} &= \exp\int_t^T \alpha_s ds.
\end{aligned}$$

Then

$$y_T=y_t-\int_t^T \alpha_s y_s ds-\int_t^T \theta_s ds-(A_T-A_t)+\int_t^T (z_s,dB_s).$$

Using Ito's formula

$$\Gamma_{t,T}y_T=\Gamma_{t,t}y_t-\int_t^T \Gamma_{t,s}(\theta_s ds+dA_s)+\int_t^T \Gamma_{t,s}(z_s,dB_s)$$

and, taking conditional expectation

$$y_t=E\left(\Gamma_{t,T}y_T+\int_t^T \Gamma_{t,s}(\theta_s ds+dA_s) \mid F_t\right) \geq 0.$$

□

Proof of Theorem 1.

Existence.

Step 1. Consider a sequence of functions $\phi_n \in C^\infty$ such that $0 \leq \phi_n \leq 1$ and

$$(3) \quad \begin{aligned} \phi_n(y) &= 1 \text{ if } |y| \leq \frac{1}{2^n}, \\ &= 0 \text{ if } |y| \geq \frac{2}{2^n}, \end{aligned}$$

and define

$$f_n(t,\omega,y,z) = f(t,\omega,y,z) + \phi_n(y - S_t)(f(t,\omega,S_t,V_t) + U_t)^-.$$

Assume first that $(f(t,\omega,S_t,V_t)+U_t)^-$ is bounded. Then f_n is Lipschitz continuous in y,z so the standard BSDE $\mathbf{Eq}(\xi,\mathbf{f}_n)$ has a unique solution (Y^n,Z^n). Since $f_n(t,\omega,S_t,V_t)+U_t \geq 0$ the hypothesis (H_4) holds true and so $Y^n \geq S$. If $(f(t,\omega,S_t,V_t)+U_t)^-$ is not bounded we define $f_n^K = f + \phi_n[(f+U_s)^- \wedge K]$ and denote by (Y_K^n, Z_K^n) the solution of $\mathbf{Eq}(\xi,\mathbf{f}_n^K)$. By the comparison theorem $Y_K^n \uparrow$ so that we may define $Y^n = \lim_K Y_K^n$. The convergence of Z_K^n follows from standard arguments (see bellow) so we may construct a solution (Y^n,Z^n) of $\mathbf{Eq}(\xi,\mathbf{f}_n)$. It is not clear that this equation has a unique solution but this is not relevant in our framework.

Let us check that, for some constant C

$$(4) \quad E\sup_{t\leq T}|Y_t^n|^2 + E\int_0^T |Z_s^n|^2\, ds \leq CE\,|\xi|^2\,, \forall n \in N,$$

Using Ito's formula, the linear growth of f and the simple inequality $|ab| \leq \frac{\alpha}{2}|a|^2 + \frac{1}{2\alpha}|b|^2$one gets

$$\begin{aligned} E\,|Y_t^n|^2 + E\int_t^T |Z_s^n|^2\, ds &\leq E\,|\xi|^2 + CE\int_t^T |Y_s^n|\,(1+|Y_s^n| + |Z_s^n|)ds \\ &\leq E\,|\xi|^2 + C'E\int_t^T (1+|Y_s^n|^2)ds + \frac{1}{2}E\int_t^T |Z_s^n|^2\, ds. \end{aligned}$$

This together with Gronwall's lemma yields $E\,|Y_t^n|^2 + E\int_t^T |Z_s^n|^2\, ds \leq C$. One then uses Burkholder's inequality in order to replace $|Y_t^n|^2$ by $\sup_{t\leq T}|Y_t^n|^2$. So (4) is proved.

Step 2. Since $f_n \downarrow$, the comparison theorem yields $Y^n \downarrow$ so we may define $Y = \lim_n \downarrow Y^n$. By the monotone convergence theorem

$$(5) \quad E\int_0^T |Y_t^n - Y_t|^2\, dt \to 0.$$

Moreover using Ito's formula, the linear growth of f_n and f_m, (4) and (5) yield

$$
\begin{aligned}
& (6) \quad E\left|Y_t^n-Y_t^m\right|^2+E\int_t^T\left|Z_s^n-Z_s^m\right|^2 ds \\
= & \quad 2E\int_t^T (Y_s^n-Y_s^m)(f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m))ds \leq \\
\leq & \quad C'E\int_t^T \left|Y_s^n-Y_s^m\right|(\left|f_n(s,Y_s^n,Z_s^n)\right|+\left|f_m(s,Y_s^m,Z_s^m)\right|)ds \rightarrow 0.
\end{aligned}
$$

So the sequence $(Z^n)_{n\in N}$ is Cauchy in L^2 and we may define $Z=\lim Z^n$.

Finally, using Burkholder's inequality one gets $E\sup_{t\leq T}\left|Y_t^n-Y_t^m\right|^2\rightarrow 0$ as $n,m\rightarrow\infty$ and so $E\sup_{t\leq T}\left|Y_t^n-Y_t\right|^2\rightarrow 0$. In particular Y is continuous.

Step 3. Denote $\alpha^n(s,\omega)=\phi_n(Y_s^n-S_s)\in L^2([0,T]\times\Omega,\mathcal{P},ds\times dP)$, where $\mathcal{P}$ designees the $\sigma-$field of the predictible processes. Since α^n are equally bounded, it follows that they are bounded in L^2 so there exists $\alpha\in L^2([0,T]\times\Omega,\mathcal{P},ds\times dP)$ such that, passing to a subsequence which we still denote by $(\alpha^n)_n$, $\alpha^n\rightharpoonup\alpha$ weakly.

Let us prove that (Y,Z,α) solves $\mathbf{Eq}(\xi,\mathbf{f},\mathbf{S})$. We define

$$
\begin{aligned}
I_n(t) &= :Y_t^n-\xi-\int_t^T f(s,Y_s^n,Z_s^n)+\alpha_s^n(f(s,S_s,V_s)+U_s)^- ds-\int_t^T Z_s^n dB_s \\
I(t) &= :Y_t-\xi-\int_t^T f(s,Y_s,Z_s)+\alpha_s(f(s,S_s,V_s)+U_s)^- ds-\int_t^T Z_s dB_s.
\end{aligned}
$$

We take $\theta\in L^2([0,T]\times\Omega,\mathcal{P},ds\times dP)$. Since $\alpha^n\rightarrow\alpha$ weakly, one has

$$
E\int_0^T\theta_s\alpha_s^n(f(s,S_s,V_s)+U_s)^- ds\rightarrow E\int_0^T\theta_s\alpha_s(f(s,S_s,V_s)+U_s)^- ds.
$$

Moreover, since $Y^n\rightarrow Y$ and $Z^n\rightarrow Z$ in L^2 we obtain

$$
E\int_0^T\theta_s I_s^n ds\rightarrow E\int_0^T\theta_s I_s ds.
$$

Since (Y^n,Z^n,α) solves $\mathbf{Eq}(\xi,\mathbf{f}_n,\mathbf{S})$, one has $I^n=0$ and so we obtain $E\int_0^T\theta_s I_s ds=0$. It folows that $I=0, ds\times dP$ almost suerly and since $s\rightarrow I_s(\omega)$ is almos suerly continuous, we obtain $I_s(\omega)=0,\forall s\in[0,T], dP(\omega)$ almost suerly. This means that (Y,Z,α) solves $\mathbf{Eq}(\xi,\mathbf{f},\mathbf{S})$.

It is clear that $Y_t=\lim_n Y^n(t)\geq S(t)$.

Step 4. Let us check that $\alpha_t=\alpha_t 1_{\{Y(t)=S(t)\}}$. If $Y(t)>S(t)$ then $Y^n(t)\geq Y(t)>S(t)$ and so $\alpha_t^n=\phi_n(Y_t^n-S_t)\rightarrow 0, a.s.$ Then

$$
E\int_0^T\alpha_t 1_{(Y(t)>S(t)\}}dt=\lim_n E\int_0^T\alpha_t^n 1_{(Y(t)>S(t)\}}dt=0.
$$

Step 5. It remains to check that $0 \leq \alpha \leq 1, ds \times dP, a.s.$ Take $\epsilon > 0$. Since $\alpha^n \leq 1$,

$$\delta_n =: E \int_0^T (\alpha_t - \alpha_t^n) 1_{(\alpha_s > 1+\epsilon)} ds \geq \epsilon E \int_0^T 1_{(\alpha_s > 1+\epsilon)} ds.$$

But, since $\alpha^n \rightharpoonup \alpha$ weakly, we know that $\delta_n \to 0$. So we have $E \int_0^T 1_{(\alpha_s > 1+\epsilon)} ds = 0, \forall \epsilon > 0$ and so the proof is completed (the proof of $\alpha \geq 0, ds \times dP, a.s.$ is similar).

Uniqueness. Follows from Lemma 5 below.□

Remark 6 *In the previous proof we have used the fact that a subsequence of $(\alpha^n)_{n \in N}$ conveges weakly to α. In fact the uniqueness of the solution of the equation* $\mathbf{Eq}(\xi, \mathbf{f}, \mathbf{S})$ *implies that $(\alpha^n)_{n \in N}$ has a single limit point so that $\alpha = \lim_n \alpha^n$ weakly.*

2.2 General obstacle

We consider now a general obstacle

$$S_t = C_t + A_t$$

where $C \in \mathcal{S}$ and A is an adapted increasing step process.

We look to the following RSDE

$$\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S}) \quad Y_t = \xi + \int_t^T f(s, \omega, Y_s, Z_s) ds + K_T - K_t - \int_t^T (Z_s, dB_s).$$

A solution of this equation is defined as a triplet $(Y, Z, K) \in (H^2)^{d+2}$ which satisfies the equation and such that $\diamondsuit \quad Y_t \geq S_t, \forall 0 \leq t \leq T, Pa.s.$ $\diamondsuit \quad Y \in \mathcal{S}$ $\diamondsuit \quad K$ is a continuous increasing process such that $\int_0^T (Y_t - S_t) dK_t = 0$.

Remark 7 *The previous study shows that, when $dS_t = V_t dB_t + U_t dt + A_t$, one has $dK_t = \alpha_t 1_{\{Y_t = S_t\}} (f(t, Y_t, Z_t) + U_t - V_t)^- dt$.*

Theorem 2 *Assume (H_1) and (H_3). Then the RBSDE* $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S})$ *has a unique solution.*

The main step in the proof is the following.

Lemma 1 *Let $(Y, Z, K), (\overline{Y}, \overline{Z}, \overline{K})$ solve* $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S})$ *(resp.* $\mathbf{E}q(\xi, \mathbf{f}, \overline{S})$*). Then*

$$(7) \quad \delta =: E \int_0^T \left| S_s - \overline{S}_s \right| d(K_s + \overline{K}_s)$$

is sufficiently small (it is choosen in the proof). Then there exists a constant C such that

$$(8) \quad E \sup_{0 \leq s \leq T} \left| K_s - \overline{K}_s \right|^2 + E \sup_{0 \leq s \leq T} \left| Y_s - \overline{Y}_s \right|^2 + \int_0^T E \left| Z_s - \overline{Z}_s \right|^2 ds \leq C\sqrt{\delta}.$$

and

$$(9) \quad E\sup_{0\le s\le T}|K_s|^2 + E\sup_{0\le s\le T}|Y_s|^2 + \int_0^T E\,|Z_s|^2\,ds \le CE(|\xi|^2 + \sup_{0\le s\le T}|S_s|^2).$$

Proof. The arguments are essentially the same as in [El-K.K.P.P.Q]. We denote $\Delta Y = Y - \overline{Y}, \Delta Z = Z - \overline{Z}, \Delta K = K - \overline{K}$ and $\Delta f_s = f(s, X_s, Y_s, Z_s) - f(s, X_s, \overline{Y}_s, \overline{Z}_s)$. They satisfies the equation

$$\Delta Y_s = \int_s^T \Delta f_r dr + \Delta K_T - \Delta K_s - \int_s^T (\Delta Z_r, dB_r).$$

By Ito's formula

$$(10) \quad |\Delta Y_s|^2 + \int_s^T |\Delta Z_r|^2\,dr = 2\int_s^T \Delta Y_r \Delta f_r dr + 2\int_s^T \Delta Y_r d\Delta K_r - 2\int_s^T \Delta Y_r(\Delta Z_r, dB_r).$$

Since K (respectively $\overline{K}$) grows when $Y = S$ (respectively $\overline{Y} = \overline{S}$) only, and $Y \ge S$ and $\overline{Y} \ge \overline{S}$,

$$\int_s^T \Delta Y_r d\Delta K_r = \int_s^T (S_r - \overline{Y}_r)dK_r + \int_s^T (\overline{S}_r - Y_r)d\overline{K}_r \le \int_s^T \left|\overline{S}_r - S_r\right| d(K_r + \overline{K}_r)$$

and therefore $E\int_s^T \Delta Y_r d\Delta K_r \le \delta$.

Using the Lipschiz continuity of f,

$$|\Delta Y_r \Delta f_r| \le C\,|\Delta Y_r|\,(|\Delta Y_r| + |\Delta Z_r|) \le C'\,|\Delta Y_r|^2 + \frac{1}{4}\,|\Delta Z_r|^2\,.$$

We take expectation in (10) and use these inequalities in order to get

$$E\,|\Delta Y_s|^2 + \frac{1}{2}E\int_s^T |\Delta Z_r|^2\,dr \le CE\int_s^T |\Delta Y_r|^2\,dr + \delta,$$

and finally, by Gronwall's lemma

$$(11) \quad E\,|\Delta Y_s|^2 + \frac{1}{2}E\int_s^T |\Delta Z_r|^2\,dr \le C\delta.$$

We shall now use Burkholder's inequality in order to get the $\sup_s$ inside the expectation. By (10) and the two inequalities which follow we get

$$\begin{aligned}(12) \quad \sup_{0\le s\le T}|\Delta Y_s|^2 \quad &\le \quad C\int_0^T |\Delta Y_r|\,(|\Delta Y_r| + |\Delta Z_r|)dr \\ &\quad +2\int_0^T \left|\overline{S}_r - S_r\right| d(K_r + \overline{K}_r) + 2\sup_{0\le s\le T}|M_s|\end{aligned}$$

with $M_s = \int_0^s \Delta Y_r (\Delta Z_r, dB_r)$. Using Burkholder's inequality first and then Schwartz inequality, one obtains

$$
\begin{aligned}
(13) \quad E \sup_{0\le s\le T} |M_s| &\le bE\left[\left(\int_0^T |\Delta Y_r|^2 |\Delta Z_r|^2\, dr\right)^{1/2}\right] \\
&\le bE\left[\left(\sup_{0\le s\le T} |\Delta Y_r|^2 \int_0^T |\Delta Z_r|^2\, dr\right)^{1/2}\right] \\
&\le b(1 + E\sup_{0\le s\le T} |\Delta Y_r|^2)\left(E\int_0^T |\Delta Z_r|^2\, dr\right)^{1/2} \\
&\le \frac{1}{4} E\sup_{0\le s\le T} |\Delta Y_r|^2 + C'\sqrt{\delta}
\end{aligned}
$$

the last inequality being a consequence of (11) and of the hypothesis (7) (one chooses δ such that $b\sqrt{C^{1/2}\delta} \le 1/4$, where C is the constant in (11)).

Now, by (11),(12) and (13) one gets

$$
(14) \quad E\sup_{0\le s\le T} |\Delta Y_s|^2 \le C\sqrt{\delta}.
$$

Coming back to the equation and using the Lipschitz continuity property of f we get

$$
\begin{aligned}
\sup_{0\le s\le T} |\Delta K_s|^2 &\le 2\sup_{0\le s\le T} |\Delta Y_s|^2 + C\int_0^T (|\Delta Y_r| + |\Delta Z_r|)^2 dr \\
&\quad +2\sup_{0\le s\le T}\left|\int_s^T (Z_s, dB_s)\right|
\end{aligned}
$$

and further, by (11), (14) and Burkholder's inequality $E\sup_{t\le s\le T} |\Delta K_s|^2 \le \sqrt{\delta}$ and the proof of (8) is completed.

The proof of (9) is similar and we skip it.$\square$

Proof of the theorem. The uniqueness follows from (8). In order to construct a solution we define $C_n = C * \phi_n$ and $S_n = C_n + A$. Since $t \to C_n(t)$ is differentiable and $|\partial_t C_n(t)| \le 2^n \sup_t |C(t)|$ the obstacle S_n satisfies the hypothesis in Theorem 1 and so we have a solution (Y_n, Z_n, K_n) for $\mathbf{E}q(\xi, \mathbf{f}, \mathbf{S}_n)$. By (8) and (9)

$$
\begin{aligned}
& E\sup_t |Y_n(t)-Y_m(t)|^2 + E\sup_t |K_n(t)-K_m(t)|^2 + E\int_0^T |Z_n(t)-Z_m(t)|^2\,dt \\
\leq\ & C\left(E\int_0^T |S_n(t)-S_m(t)|\,d(K_n(t)+K_m(t))\right)^{1/2} \\
\leq\ & C\left(E\sup_t |S_n(t)-S_m(t)|^2\right)^{1/4} (E(|K_n(T)|+|K_m(T)|))^{1/4} \\
\leq\ & C'\left(E\sup_t |S_n(t)-S_m(t)|^2\right)^{1/4} \to 0 \text{ as } n,m\to\infty.
\end{aligned}
$$

So we may set $Y=\lim_n Y_n$, $Z=\lim_n Z_n$ and $K=\lim_n K_n$, and, by passing to the limit in $\mathbf{Eq}(\xi,\mathbf{f},\mathbf{S}_n)$ we check that (Y,Z,K) solve $\mathbf{E}q(\xi,\mathbf{f},\mathbf{S})$.$\square$

3 Relation with semilinear reflected PDE's

3.1 Regular obstacle

In this section we consider the Markovian framework. Let us introduce some notation. $\rho: R^N \to R_+$ is a continuous, strictly positive function which is fixed in the sequel and represents a weight. We assume that $\int \rho(x)dx < \infty$. A typical example would be $\rho(x)=(1+|x|)^{-\beta}, \beta>1$. We shall work with functions in the weighted L^2-space $L^2_\rho =: L^2(R^N,\rho(x)dx)$.

The objects comming on in this section are the following.

$\diamondsuit$ The infinitesimal operator

$$
\begin{aligned}
L\phi(x) &= \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^*)^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}\phi(x) + \sum_{i=1}^n b^i(x)\frac{\partial}{\partial x^i}\phi(x) \\
&= \sigma^*\nabla(\sigma^*\nabla\phi) + \widetilde{b}\nabla\phi
\end{aligned}
$$

where

$$
(15.i) \quad \sigma \in C_b^3(R^N;R^{N\times d}), b\in C_b^2(R^N;R^N)
$$

and $\widetilde{b} =: \frac{1}{2}\sigma^*\nabla\sigma + b$.

$\diamondsuit$ The weighted Sobolev (actually Dirichlet) space

$$
\begin{aligned}
H_\rho^1 &= \{u:[0,T]\times R^N \to R, u_t\in L^2_\rho, \forall 0\leq t\leq T, \\
\sigma^*\nabla u &\in L^2([0,T]\times R^N, dt\times\rho(x)dx)\}
\end{aligned}
$$

$\diamondsuit$ The diffusion process with infinitesimal operator L, which starts from x at time t

$$
X_s^{t,x} = x + \sum_{j=1}^d \int_t^s \sigma^{ij}(X_r^{t,x})dB^j(r) + \int_t^s b^i(X_r^{t,x})dr, \quad t\leq s\leq T.
$$

$\diamondsuit$ The obstacle $h + q$ with

$$\begin{array}{rcl} (15.ii) \quad h & \in & C^{1,2}[0,T] \times R^N, R), \text{ such that } |h(x)| \leq C(1+|x|^\beta), \\ h(t,.) & \in & L^2_\rho, \forall 0 \leq t \leq T, \text{ and } \partial_t h, \partial_x h, \partial_x^2 h \in L^2(dt \times \rho(x)dx) \end{array}$$

$$\begin{array}{rcl} (15.iii) \quad q & : & [0,T] \times R^N \to R, \text{ is measurable, } |q(x)| \leq C(1+|x|^\beta), \text{ and for every } t,x \\ s & \to & q(s, X_s^{t,x}) \text{ is a square integrable, càdlàg submartingale.} \end{array}$$

Here β is a strictly positive number which is fixed in the sequel.

$\diamondsuit$ The final condition

$$(15.iv) \quad g \in L^2_\rho, g \geq h_T + q_T.$$

$\diamondsuit$ The driver $f : [0,T] \times R^N \times R \times R^d \to R$ which is a measurable function such that:

$$\begin{array}{rcl} (15.v) \quad \displaystyle\int_{R^N} \rho(x) \left| f(t,x,0,0) \right|^2 dx & < & \infty, \\ (15.vi) \quad |f(t,x,y,z) - f(t,x,y',z')| & \leq & C(|y-y'| + |z-z'|) \end{array}$$

$\diamondsuit$ The semilinear PDE

$$\begin{array}{rcl} \mathbf{PDE(g,f,h,q)} \quad (\partial_t + L)u(t,x) + F(t,x,\alpha(t,x),u,\sigma^*\nabla u) & = & 0, \forall 0 \leq t \leq T, \\ u(T,x) & = & g(x), u \geq h, \end{array}$$

where

$$\begin{array}{rcl} F(t,x,a,y,z) & = & : f(t,x,y,z) \\ & & + a 1_{\{y=(h+q)(t,x)\}}(f(t,x,h(t,x),\sigma\nabla h(t,x)) + (\partial_t + L)h(t,x))^-. \end{array}$$

We write this equation in the variational form:

$$\begin{array}{rl} & \mathbf{PDE(g,f,h,q)} \\ & \displaystyle\int_t^T \int_{R^N} u \partial_s \varphi dx ds - \int_{R^N} g(x)\varphi(T,x) - u(t,x)\varphi(t,x) dx \\ & + \displaystyle\int_t^T \int_{R^N} (\sigma^* \nabla u)(\sigma^* \nabla \varphi) + u \nabla(\widetilde{b}\varphi) dx ds \\ = & \displaystyle\int_t^T \int_{R^N} \varphi F(s,x,\alpha,u,\sigma^*\nabla u) dx ds, \quad \forall \varphi \in C_c^{1,1}(R^N; R) \end{array}$$

Definition 1 *A weak solution of the* $\mathbf{PDE(g,f,h,q)}$ *is a pair of functions* $(u,\alpha) \in H^1_\rho \times L^2([0,T] \times R^N, dt \times \rho(x)dx)$ *which satisfies the equation* $\mathbf{PDE(g,f,h,q)}$ *and such that* $u(t,x) \geq h(t,x) + q(t,x), \forall (t,x) \in [0,T] \times R^N$.

Remark 8 *We may replace the function F in the above definition by*

$$\begin{aligned}\widetilde{F}(t,x,a,y,z) &= : f(t,x,y,z) \\ &\quad + a1_{\{y=(h+q)(t,x)\}}(f(t,x,y,z)+(\partial_t+L)h(t,x))^-\end{aligned}$$

and the equation remains unchanged (see the remark after the equation (2)*).*

$\diamondsuit$ The RBSDE

$$\begin{aligned} & \mathbf{Eq}^{t,x}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q}) \\ Y_s^{t,x} &= g(X_T^{t,x})+\int_s^T F(r,\alpha_r^{t,x},X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr-\int_s^T (Z_r^{t,x},dB_r)\end{aligned}$$

Remark 9 *This equation coincides with the RBSDE $\mathbf{Eq}(\mathbf{g}(\mathbf{X}_T^{t,x}),\mathbf{f}_{t,x},\mathbf{S}_{t,x})$ (see (2)) where $f_{t,x}(s,\omega,y,z)=f(s,X_s^{t,x}(\omega),y,z)$ and $S_{t,x}(s,\omega)=(h+q)(s,X_s^{t,x}(\omega))$. Using Doob's Mayer decomposition first and the reresentation theorem then we may write $q(s,X_s^{t,x}(\omega))=\int_t^s(\theta_r^{t,x},dB_r)+\int_t^s\lambda_s^{t,x}drA_s^{t,x}$ where $\theta_s^{t,x}$ is a square integrable process,$\lambda_s^{t,x}$ is an integrable process and $A_s^{t,x}$ is an adapted non decreasing process such that $dA_s^{t;x}$ is singular with respect to the Lebesgue measure. With the notation in the previous section, we have $V_s=\theta_r^{t,x}+\sigma\nabla h(r,X_r^{t,x})$ and $U_s=\lambda_r^{t,x}+(\partial_s+L)h(s,X_s^{t,x})$ and $A_s=A_s^{t,x}$.*

In this section we will prove existance and uniqueness for $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ and we will establisch the probabilistic interpretation of the solutions in terms of $RBSDE's$. Our baisic idea is to employ existence and uniqueness for the solutions of $RBSDE's$, and the a priory inequalyties given in Lemma 4, in order to do it. In order to perform this program we shall use the following equivalence of norms lemma (see [B.M], Appendix B or [Ba.L],[K.1]): there exists two constants $0<c<C<\infty$ such that, under the regularity assumptions $(15,i)$, for every $0\le t\le s\le T$ and every measurable $\phi\ge 0$

$$\begin{aligned}(EN)\quad c\int_0^T\int_{R^N}\phi(s,x)\rho(x)dxds &\le \int_0^T\int_{R^N}E\phi(s,X_s^{t,x})\rho(x)dxds \\ &\le C\int_0^T\int_{R^N}\phi(s,x)\rho(x)dxds.\end{aligned}$$

This inequalities are crucial because they permit to translate a priory estimates for $RBSDE's$ in terms of $PDE's$ (and the convers as well). So they represent somehow a dictionary which allows to pass from the stochastic to the deterministic point of view.

Theorem 3 *There exists a unique solution (u,α) of the equation $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$. This solution is related to the solution $(Y_s^{t,x},Z_s^{t,x},\alpha_s^{t,x})_{t\le s\le T}$ of the RBSDE $\mathbf{Eq}^{t,x}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ by*

$$Y_s^{t,x}=u(s,X_s^{t,x}),Z_s^{t,x}=\sigma^*\nabla u(s,X_s^{t,x}),\alpha_s^{t,x}=\alpha(s,X_s^{t,x}).$$

Proof.

Existence. For each fixed $(t,x) \in [0,T]\times R^N$ we consider the solution $(Y_s^{t,x}, Z_s^{t,x}, \alpha_s^{t,x})_{t\le s\le T}$ of the reflected BSDE $\mathbf{Eq}(\mathbf{g}(\mathbf{X}_T^{t,x}), \mathbf{f}_{t,x}, \mathbf{S}_{t,x})$. We shall first prove that there exists some measurable functions u, v, α such that

$$Y_s^{t,x} = u(s, X_s^{t,x}), Z_s^{t,x} = v(s, X_s^{t,x}) \text{ and } \alpha_s^{t,x} = \alpha(s, X_s^{t,x}) \quad ds \times \rho(x)dx \times dP, a.s.$$

The proof concerning Y and Z is a strightforward application of the equivalence of norms result. We use the same approximation method as in the proof of Theorem 1. We consider the functions

$$f_n(s,\omega,y,z) = f(s, X_s^{t,x}, y, z) - \phi_n(y - S_s^{t,x})\theta(s, X_s^{t,x}))^-$$

with $\theta(t,x) = (f(t,x,h(t,x), \sigma\nabla h(t,x)) + (\partial_t + L)h(t,x))^-$, and we define $(Y_{n,s}^{t,x}, Z_{n,s}^{t,x})_{t\le s\le T}$ to be the solution of the standard BSDE $E(g(X_T^{t,x}), f_n)$. We already know (see (6)) that for each x

$$(16) \quad E \sup_{t\le s\le T} \left|Y_{n,s}^{t,x} - Y_s^{t,x}\right|^2 + E\int_t^T \left|Z_{n,s}^{t,x} - Z_s^{t,x}\right|^2 ds \to 0.$$

We also know (see (4)) that

$$E \sup_{t\le s\le T} \left|Y_{n,s}^{t,x}\right|^2 + E\int_t^T \left|Z_{n,s}^{t,x}\right|^2 ds \le CE\left|g(X_T^{t,x})\right|^2.$$

Using the second inequality in (EN) we get

$$\int \rho(x)E\left|g(X_T^{t,x})\right|^2 dx \le C\int \rho(x)\left|g(x)\right|^2 dx < \infty$$

and so, using (4) and (16)

$$\int \rho(x)E \sup_{t\le s\le T} \left|Y_{n,s}^{t,x} - Y_s^{t,x}\right|^2 dx + \int \rho(x)E\int_t^T \left|Z_{n,s}^{t,x} - Z_s^{t,x}\right|^2 dsdx \to 0.$$

Denote $u_n(t,x) = Y_{n,t}^{t,x}$. It is known (see [B.M]) that $u_n \in H_\rho^1$, $Y_{n,s}^{t,x} = u_n(s, X_s^{t,x})$ and $Z_{n,s}^{t,x} = v_n(s, X_s^{t,x})$ where $v_n = \sigma^*\nabla u_n$. Using the equivalence of norms

$$\begin{aligned} & \int \rho(x)\left|(u_m - u_n)(t,x)\right|^2 dx + \int_t^T\int \rho(x)\left|(v_m - v_n)(s,x)\right|^2 dxds \\ \le \quad & C\int \rho(x)E\left|Y_{m,s}^{t,x} - Y_{n,s}^{t,x}\right|^2 dx + C\int \rho(x)E\int_t^T \left|Z_{m,s}^{t,x} - Z_{n,s}^{t,x}\right|^2 ds \to 0 \end{aligned}$$

so we may define $u = \lim_n u_n$ and $v = \lim_n v_n$, the limits being taken in $L^2(\rho(x)dx)$ (resp. in $L^2(ds \times \rho(x)dx)$). We also have $u(t,x) = Y_t^{t,x} \ge S_t^{t,x} = (h+q)(t,x)$.

By the equivalence of norms again

$$\begin{aligned}
& \int \rho(x) E \left| Y_s^{t,x} - u(s, X_s^{t,x}) \right|^2 dx \\
\leq \quad & 2 \int \rho(x) E \left| Y_s^{t,x} - Y_{n,s}^{t,x} \right|^2 dx + 2 \int \rho(x) E \left| u_n(s, X_s^{t,x}) - u(s, X_s^{t,x}) \right|^2 dx \\
\leq \quad & 2 \int \rho(x) E \left| Y_s^{t,x} - Y_{n,s}^{t,x} \right|^2 dx + C \int \rho(x) E \left| u_n(s,x) - u(s,x) \right|^2 dx \to 0
\end{aligned}$$

so that $Y_s^{t,x} = u(s, X_s^{t,x}), dx \times dPa.s.$ In the same way one proves that $Z_s^{t,x} = v(s, X_s^{t,x}), ds \times \rho(x)dx \times dP, a.s.$ Since $u_n \to u$ and $\sigma \nabla u_n \to v$ standard arguments (see [B.M] for details) give $u \in H_\rho^1$ and $\sigma \nabla u_n = v$.

Let us now construct α. Let $\alpha_{n,s}^{t,x} = \phi_n(Y_{n,s}^{t,x} - S_{n,s}^{t,x})$. From the remark after the proof of Theorem 1 we know that, for each fixed x, $\alpha_s^{t,x} = \lim_n \alpha_{n,s}^{t,x}$ the limit being a weak limit in $L^2(ds \times dP)$. It is known (see Dunford and Schwarz [D.S] or Ekland and Temam [E.T]) that one may construct convex combinations $\beta_{n,s}^{t,x} = \sum_1^{k_n} \lambda_k^n \alpha_{k,s}^{t,x}$ such that $\alpha_s^{t,x} = \lim_n \beta_{n,s}^{t,x}$, the limit being now a strong limit i.e. in $L^2(ds \times dP)$.

On the other hand $\alpha_{n,s}^{t,x} = \phi_n((u_n - (q+h))(s, X_s^{t,x}))$ so that $\alpha_{n,s}^{t,x} = \alpha_n(s, X_s^{t,x})$ with $\alpha_n(t,x) = \alpha_{n,t}^{t,x}$ (one uses the flow property for X). It follows that $\beta_{n,s}^{t,x} = \beta_n(s, X_s^{t,x})$ with $\beta_n(t,x) = \sum_1^{k_n} \lambda_k^n \alpha_{n,k} = \beta_{n,t}^{t,x}$. We use the equaivalence of norms (this is why we need strong convergence instead of weak convergence) in order to get

$$\int_0^T \int \left| (\beta_n - \beta_m)(s,x) \right|^2 \rho(x) dx \leq CE \int_0^T \int \left| (\beta_n - \beta_m)(s, X_s^{0,x}) \right|^2 \rho(x) dx.$$

Since $(\beta_{n,s}^{t,x})_n$ is a Cauchy sequence in $L^2(ds \times \rho(x)dx \times dP)$ it follwos that $(\beta_n(s,x))_n$ is a Cauchy sequence in $L^2(ds \times \rho(x)dx)$ and so we may define $\alpha = \lim_n \beta_n$. The same reasoning as above shows that $\alpha_s^{t,x} = \alpha(s, X_s^{t,x}), dx \times ds \times dPa.s.$

So, we know now that $(Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T}$ solves the standard BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T \overline{f}(r, X_r^{t,x}) dr - \int_s^T (Z_r^{t,x}, dB_r)$$

with $\overline{f}(r,x) =: F(r, x, \alpha(r,x), u(r,x), v(r,x))$. In particular, it is proved in Bally, Matoussi [B.M] that $u \in H_\rho^1$, $v = \sigma^* \nabla u$ and u solves $\mathbf{PDE(g, f, h, q)}$.

Uniqueness. The proof is given in a more general frame: see Theorem 7. iii)□

We close this section with a the remark that the obstacle problem discussed above represents an extension of the classical maximum principle. Let us be more precise. We consider two functions $f : [0,T] \times R^d \times R \times R^d \to R$ and $g : R^d \to R$ and we denote by u^F the solution (in variational sense) of the PDE

$$(\partial_t + L) u^f + f(t, x, u^f, \sigma \nabla u^f) = 0, \quad u^f(T, x) = g(x).$$

The classical maximum principle asserts that (under appropiate hypothesis) for every $h \in C^{1,2}([0,T] \times R^d)$, if

$$g(x) \geq h(T,x) \quad and \quad (\partial_t + L)h + f(t,x,h,\sigma\nabla h) \geq 0 \Rightarrow u^f \geq h.$$

In particular, using the comparison theorem, this implies that for every $F \geq f$, $u^F \geq u^f \geq h$. So we may rephrase the above assertion as

$$g(x) \geq h(T,x) \quad and \quad (\partial_t + L)h + f(t,x,h,\sigma\nabla h) \geq 0 \Rightarrow u^F \geq h, \forall F \geq f.$$

We come now back to our obstacle problem and denote by $\alpha_{f,h}$ the function which appears in the solution of $PDE(g,f,h)$. Then we may drop out the restriction $(\partial_t + L)h + f(t,x,h,\sigma\nabla h) \geq 0$ and rephraze the above assertion as

$$\begin{aligned} g(x) &\geq h(T,x) \quad \Rightarrow \quad u^F \geq h \\ \forall F(t,x,y,z) &\geq f(t,x,y,z) + \alpha_{f,h}(x) 1_{(y=h(t,x))}((\partial_t + L)h + f(t,x,h,\sigma\nabla h))^-. \end{aligned}$$

In order to prove this we denote

$$\widetilde{f}(t,x,y,z) = f(t,x,y,z) + \alpha_{f,h}(x) 1_{(y=h(t,x))}((\partial_t + L)h + f(t,x,h,\sigma\nabla h))^-$$

and we note that, in view of our theorem, we have $\widetilde{u} \geq h$, where $\widetilde{u}$ is the solution of $PDF(g,\widetilde{f},h,0)$. Then, using the comparison theorem it follows that $u^F \geq \widetilde{u} \geq h$. So we obtain

Corollary 1 *(Maximum principle) Suppose that f satisfies* (15). *Then, for every $h \in C^{1,2}([0,T] \times R^d)$ there exists a function $\alpha_{f,h} = \alpha_{f,h}(x) \in [0,1]$ such that*

$$\begin{aligned} g(x) &\geq h(T,x) \quad \Rightarrow \quad u^F \geq h \\ \forall F(t,x,y,z) &\geq f(t,x,y,z) + \alpha_{f,h}(x) 1_{(y=h(t,x))}((\partial_t + L)h + f(t,x,h,\sigma\nabla h))^-. \end{aligned}$$

Remark 10 *It is ruther natural to think that the function $\alpha_{f,h}$ produced by the obstacle problem is, in some sense, the minimal weight for $1_{(y=h(t,x))}((\partial_t + L)h + f(t,x,h,\sigma\nabla h))^-$ which permits to obtain the above inequality for every F. But it is not clear for us how to prove it.*

3.2 General obstacle

The aim of this section is to generalize the results presented above for an obstacle h which is not regular anymore. We just assume that h is a continuous function. As a consequence, there exists a sequence $h_n \in C^{1,2}([0,T] \times R^N), n \in N$, such that $h_n(t,x) \to h(t,x), dt \times dx, a.s., |h_n(t,x)| \leq C(1+|x|)^\beta$ and, for every $t \in [0,T]$

$$(17) \quad E(\sup_{t \leq s \leq T} \left|(h_m - h_n)(s, X_s^{t,x})\right|^2) \to 0 \text{ as } n,m \to \infty, dx, a.s.$$

Remark 11 *In the following the only thing we shall use is (17) which in fact holds true even if h is not continuous (for example if h is in H^1 as a function of x). In order to simplify the exposition we live out this kind of generalizations. They are presented in the preprint [B.C.F].*

Since $\alpha(f+(\partial_t+L)h)^- dxds$ does not make sense anymore we have to replace it by an abstract measure $\mu(dx,ds)$. More precisely we are interested in the PDE

$$\begin{aligned}
&\mathbf{PDE(g,f,h,q)}\\
&\int_t^T\int_{R^N} u\partial_s\varphi dxds-\int_{R^N}(g(x)\varphi(T,x)-u(t,x)\varphi(t,x))dx\\
&+\int_t^T\int_{R^N}((\sigma^*\nabla u)(\sigma^*\nabla\varphi)+u\nabla(\widetilde{b}\varphi))dxds\\
=\ &\int_t^T\int_{R^N}\varphi f(s,x,u,\sigma^*\nabla u)dxds+\int_t^T\int_{R^N}\varphi(t,x)1_{(u=h+q)}d\mu(dx,ds),\\
\forall\varphi\ \in\ &C_c^{1,1}(R^N,R), 0\le t\le T.
\end{aligned}$$

The solution of the above equation is a couple (u,μ) which satisfy $\mathbf{PDE(g,f,h,q)}$ with $u\in H_\rho^1$ such that $u\ge h+q$, $ds\times dx, a.s.$ and μ a positive measure such that $\int_0^T\int\rho(x)d\mu(t,x)<\infty$.

Remark 12 *This may also be written as*

$$(\partial_t+L)u+f(t,x,u,\sigma^*\nabla u)=-1_{\{u=h+q\}}\mu,\quad u\ge h+q,\quad u_T=g$$

the equality being understood in distribution sense. Since μ is a positive measure, this may be expressed as

$$\begin{aligned}
(\partial_t+L)u+f(t,x,u,\sigma^*\nabla u)&\le 0,\quad u\ge h+q,\quad u_T=g,\\
(\partial_t+L)u+f(t,x,u,\sigma^*\nabla u)&= 0\ \text{on}\ u>h+q.
\end{aligned}$$

which is exactly the formulation given for the obstacle problem by Bensoussan and Lions in [Be.L]. Our definition contains one more regularity requirement, namely that $(\partial_t+L)u+f(t,x,u,\sigma^*\nabla u)$ is a measure. Such properties are also discussed in [Be.L] Ch.3, 2.10.

The above PDE still corresponds to a RBSDE but in this case we have to consider a general increasing process K that pushes up in order to et $Y\ge S$, as in [El-K.K.P.P.Q]:

$$\mathbf{Eq}^{t,x}\mathbf{(g,f,h,q)}\quad Y_s^{t,x}=g(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+K_T^{t,x}-K_s^{t,x}-\int_s^T(Z_r^{t,x},dB_r).$$

The solution of this RBSDE is a triplet $(Y_s^{t,x},Z_s^{t,x},K_s^{t,x})_{t\le s\le T},(t,x)\in[0,T]\times R^n$, such that (18)

i) $(t,s,x,\omega) \to (Y_s^{t,x}(\omega), Z_s^{t,x}(\omega), K_s^{t,x}(\omega))$ is measurable and for each fixed t, x $(s,\omega) \to (Y_s^{t,x}(\omega), Z_s^{t,x}(\omega), K_s^{t,x}(\omega))$ is a progressively measurable process.

$$\text{ii)} \quad \int_{R^n} \rho(x)(E\left|K_T^{t,x}\right|^2 + \int_t^T (E\left|Y_s^{t,x}\right|^2 + E\left|Z_s^{t,x}\right|^2)ds)dx < \infty.$$

For every almost $x \in R^N$
iii) $s \to K_s^{t,x}$ is a continuous, non decreasing process and $K_t^{t,x} = 0$,
iv) $Y_s^{t,x} \geq S_s^{t,x}, \quad \forall t \leq s \leq T, dPa.s.$,where $S_s^{t,x} = (h+q)(s, X_s^{t,x})$,
v) $\int_t^T (Y_s^{t,x} - S_s^{t,x})dK_s^{t,x} = 0, dPa.s. \forall t \leq T$
vi) $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$ satifies $\mathbf{Eq}^{t,x}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$, for every $t \leq T$.

Remark 13 *For each fixed (t,x) the equation $\mathbf{Eq}^{t,x}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$ is a RBSDE with obstacle $S_s^{t,x}$, so, if $h+q$ is a continuous function we are in the framework of [El-K.K.P.P.Q]. What is different here is that $s \to q(s, X_s^{t,x})$ may have positive jumps, so $s \to S_s^{t,x}$ is generally not continuous.*

We need some more notation. It is well known (see [K1] or [I.W] for example) that one may choose a version of the stochastic flow $x \to X_s^{t,x}$ which is differentiable with respect to x and invertible. We denote by $\widehat{X}_s^{t,x}$ the inverse of $x \to X_s^{t,x}$ and we also denote by $J(\widehat{X}_s^{t,x})$ is the determinant of the Jacobian matrix of $\widehat{X}_s^{t,x}$.

The main result in this section is the following.

Theorem 4 *We assume (15) except for (15.ii) and instead we assume that h is continuous. We also assume that $\rho(x) = (1+|x|)^{-p}$ with $p \geq \beta + 2$.*

i) For each (t,x) there exists a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$ of the RBSDE $\mathbf{Eq}^{t,x}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$.

ii) There exists a solution (u,μ) of $\mathbf{PDE}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$ such that

$$\begin{aligned}
(19) \quad a) Y_s^{t,x} &= u(s, X_s^{t,x}), Z_s^{t,x} = \sigma^* \nabla u(s, X_s^{t,x}), \\
& b) \int_{R^N} \int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) \psi(s,x) 1_{\{u=h+q\}}(s,x) d\mu(s,x) \\
&= \int_{R^N} \int_t^T \phi(s,x)\psi(s, X_s^{t,x}) dK_s^{t,x} dx, Pa.s.
\end{aligned}$$

for every measurable, bounded and positive functions ϕ and ψ.

iii) Let $(\overline{u}, \overline{\mu})$ be another solution of $\mathbf{PDE}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$ such that $\overline{\mu}$ satifies the property (19)b) with a certain process $\overline{K}$ instead of K. We assume that $\overline{K}$ satifies (18)i),ii),iii). Then $\overline{u} = u$ and $\overline{\mu} = \mu$.

Remark 14 *The relation (19) a) represents the satndard link betwen the solution of a PDE and the solution of BSDE's but (19) b) seems to be new. It gives the analogues probabilistic interpretation of the measure μ which appears in the obstacle problem.*

Proof. Existence and uniqueness for the solution of the RBSDE $\mathbf{Eq}^{t,x}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ is a consequence of Theorem 3.

Existence. In order to construct a solution of $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ we shall use an approximation method. Let $h_n \in C^{1,2}([0,T]\times R^N), n\in N$, be the sequence of functions given in hypothesis (17). Consider $(Y^{t,x}_{n,s}, Z^{t,x}_{n,s}, \alpha^{t,x}_{n,s})_{t\le s\le T}$, the solution of the RBSDE $\mathbf{Eq}^{t,x}(\mathbf{g},\mathbf{f},\mathbf{h}_n,\mathbf{q})$ and (u_n,α_n), the solution of $\mathbf{PDE}(\mathbf{f},\mathbf{g},\mathbf{h}_n,\mathbf{q})$, in the sense given in the previous section. We already know that

$$Y^{t,x}_{n,s} = u_n(s,X^{t,x}_s), Z^{t,x}_{n,s} = \sigma^*\nabla u_n(s,X^{t,x}_s) \text{ and } \alpha^{t,x}_{n,s} = \alpha_n(s,X^{t,x}_s).$$

We denote

$$\begin{aligned}\gamma_n(t,x) &= 1_{\{u_n=h_n+q_n\}}(t,x)\alpha_n(t,x)(f(t,x,u_n(t,x),\sigma^*\nabla u_n(t,x)) + (\partial_t+L)h_n(t,x))^- \\ &\text{and} \\ K^{t,x}_{n,s} &= \int_t^s \gamma_n(r,X^{t,x}_r)dr.\end{aligned}$$

So $(Y^{t,x}_{n,s}, Z^{t,x}_{n,s}, K^{t,x}_{n,s})_{t\le s\le T}$ satifies the equation $\mathbf{Eq}^{t,x}(\mathbf{g},\mathbf{f},\mathbf{h}_n,\mathbf{q})$. In view of (8) we know that

$$\begin{aligned}(20)\quad & E\sup_{t\le s\le T}\left|K^{t,x}_{n,s}-K^{t,x}_{m,s}\right|^2 + E\sup_{t\le s\le T}\left|Y^{t,x}_{n,s}-Y^{t,x}_{m,s}\right|^2 + E\int_t^T \left|Z^{t,x}_{n,s}-Z^{t,x}_{m,s}\right|^2 ds \\ \le\ & C\left(E\int_t^T \left|S^{t,x}_{n,s}-S^{t,x}_{m,s}\right| d(K^{t,x}_{n,s}+K^{t,x}_{m,s})\right)^{1/2} \\ \le\ & C\left(E\sup_{t\le s\le T}\left|S^{t,x}_{n,s}-S^{t,x}_{m,s}\right|^2\right)^{1/4}\times\left(E\sup_{t\le s\le T}\left|K^{t,x}_{n,s}+K^{t,x}_{m,s}\right|^2\right)^{1/4}.\end{aligned}$$

Using (9) and the growth condition on h_n+q , we get

$$\begin{aligned}E\left|K^{t,x}_{n,T}\right|^2 &\le C(E(\left|g(X^{t,x}_T)\right|^2 + \sup_{t\le s\le T}\left|S^{t,x}_s\right|^2) \\ &\le C(1+E(\left|g(X^{t,x}_T)\right|^2 + \sup_{t\le s\le T}\left|X^{t,x}_s\right|^{2\beta}) \le C(1+E\left|g(X^{t,x}_T)\right|^2 + |x|^{2\beta}).\end{aligned}$$

In particular, using the equivalence of norms (EN) and the fact that $g\in L^2_\rho$ we obtain

$$(21)\quad \sup_n E\left|K^{t,x}_{n,T}\right|^2 \le C(1+|x|^{2\beta}).$$

By (17) $\lim_n E\sup_{t\le s\le T}\left|S^{t,x}_{n,s}-S^{t,x}_{m,s}\right|^2 = 0$ and so

$$(22)\quad E\sup_{t\le s\le T}\left|K^{t,x}_{n,s}-K^{t,x}_{m,s}\right|^2 + E\sup_{t\le s\le T}\left|Y^{t,x}_{n,s}-Y^{t,x}_{m,s}\right|^2 + E\int_t^T\left|Z^{t,x}_{n,s}-Z^{t,x}_{m,s}\right|^2 ds \to 0.$$

So we may define $Y = \lim_n Y_n, Z = \lim_n Z_n$ and $K = \lim_n K_n$. It is not hard to check that they satisfy (18), so that solve the RBSDE and so i) is proved.

Let us now solve $\mathbf{PDE}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$. We first observe that in view of (20) and (21) the convergence in (22) holds in L^2_ρ as well. Since $Y^{t,x}_{n,t} = u_n(t,x)$ and $Z^{t,x}_{n,s} = \sigma^* \nabla u_n(s, X^{t,x}_s)$ we may use the equivalence of norms (see (EN)) and (22) in order to get

$$\begin{aligned} & \int \rho(x) \left| u_n(t,x) - u_m(t,x) \right|^2 dx \\ & + \int \int_t^T \rho(x) E \left| \sigma^* \nabla u_n(s,x) - \sigma^* \nabla u_m(s,x) \right|^2 ds dx \\ \leq \; & \int \rho(x) E \left| Y^{t,x}_{n,t} - Y^{t,x}_{m,t} \right|^2 dx + CE \int \rho(x) \int_t^T \left| Z^{t,x}_{n,s} - Z^{t,x}_{m,s} \right|^2 ds dx \to 0. \end{aligned}$$

So the sequence $(u_n)_n$ is Cauchy in H^1_ρ and we may define $u = \lim_n u_n$. Moreover $u \in H^1_\rho$ and so $\sigma^* \nabla u \in L^2(dt \times \rho(x) dx)$.

Let us now construct the measure μ. This is the difficult point and will be done in several steps.

Step 1. We denote $\mu_n(dt, dx) = \gamma_n(t,x) dt dx$ and $\nu_n = \rho \mu_n$, and prove that

$$(23) \qquad \sup_n \nu_n([0,T] \times R^N) < \infty.$$

Using the equivalence of norms (see (EN)) and (21), we get

$$\begin{aligned} \nu_n([0,T] \times R^N) &= \int \int_0^T \rho(x) \gamma_n(t,x) dt dx \leq C \int \int_0^T \rho(x) E \gamma_n(t, X^{0,x}_t) dt dx \\ &= C \int \rho(x) E \left| K^{0,x}_{n,T} \right| dx \leq C \int \rho(x) (1 + |x|)^\beta dx < \infty. \end{aligned}$$

Step 2. We recall that $\rho(x) = (1 + |x|)^{-p}, p \geq \beta + 2$. We prove that

$$(24) \qquad \int \frac{1}{\rho(x)} \left(E \sup_{t \leq r \leq T} \left| \rho(X^{0,x}_r) \right|^4 \right)^{1/2} dx < \infty.$$

Since $|\rho(x)| \leq 1$ we have

$$\begin{aligned} E \sup_{0 \leq r \leq T} \left| \rho(X^{t,x}_r) \right|^4 &\leq E(\sup_{0 \leq r \leq T} \left| \rho(X^{t,x}_r) \right|^4 1_{\{\sup_{0 \leq r \leq T} |X^{t,x}_r - x| \leq \frac{|x|}{2}\}}) + P(\sup_{0 \leq r \leq T} \left| X^{t,x}_r - x \right| \geq \frac{|x|}{2}) \\ &= : A(x) + B(x). \end{aligned}$$

If $\sup_{0 \leq r \leq T} \left| X^{0,x}_r - x \right| \leq \frac{|x|}{2}$ then $\left| X^{0,x}_r \right| \geq \frac{|x|}{2}$ and so $\left| \rho(X^{0,x}_r) \right| \leq (1 + \frac{|x|}{2})^{-p}$. It follows that $A(x) \leq (1 + \frac{|x|}{2})^{-4p}$ and so $\int (1 + |x|)^p A(x)^{1/2} dx < \infty$. On the other hand, if $|x| \geq 4 \left\| b \right\|_\infty T$ then (see eg. Ikeda,Watanabe [I.W])

$$B(x) \leq P(\sup_{0 \leq s \leq T} \left| \int_0^s \sigma(X^{0,x}_r) dB_r \right| \geq \frac{|x|}{4}) \leq C \exp(-C' |x|^2)$$

and so $\int(1+|x|)^p B(x)^{1/2}dx<\infty$.

Step 3. We shall prove that for every $\epsilon>0$ there exists some constant K such that

$$\int_0^T\int 1_{\{|x|\geq 2K\}}d\nu_n(s,x)\leq\epsilon,\forall n\in N. \tag{25}$$

We we write

$$\begin{aligned}\int_0^T\int 1_{\{|x|\geq 2K\}}d\nu_n(s,x) &= \int_0^T\int 1_{\{|x|\geq 2K\}}\left(1_{\{\left|\widehat{X}_s^{0,x}(\omega)-x\right|\leq K\}}+1_{\{\left|\widehat{X}_s^{0,x}(\omega)-x\right|\geq K\}}\right)d\nu_n(s,x)\\ &= :I_K^n(\omega)+L_K^n(\omega).\end{aligned}$$

This equality holds true for every ω. Taking expectation

$$\int_0^T\int 1_{\{|x|\geq 2K\}}d\nu_n=EI_K^n+EL_K^n.$$

By (23), for $K\geq 2\|b\|_\infty T$

$$\begin{aligned}EL_K^n &\leq \int_0^T\int P(\sup_{t\leq r\leq T}\left|\widehat{X}_r^{0,x}-x\right|\geq K)d\nu_n(ds,dx)\\ &\leq C\exp(-C'K^2)\nu_n([0,T]\times R^N)\leq C''\exp(-C'K^2)\end{aligned}$$

and so $EL_K^n\leq\varepsilon$ for K sufficiently large.

On the other hand, if $|x|\geq 2K$ and $\left|\widehat{X}_s^{0,x}-x\right|\leq K$ then $\left|\widehat{X}_s^{0,x}\right|\geq K$. Therefore

$$\begin{aligned}EI_K^n &\leq E\int_0^T\int 1_{\{\left|\widehat{X}_s^{0,x}\right|\geq K\}}d\nu_n(s,x)\\ &= E\int_0^T\int 1_{\{\left|\widehat{X}_s^{0,x}\right|\geq K\}}\rho(x)\alpha_n(s,x)(f+(\partial_s+L)h_n)^-(s,x)dsdx\end{aligned}$$

which, by the change of variable $y=\widehat{X}_s^{0,x}$, becomes

$$\begin{aligned}&E\int_0^T\int 1_{\{|y|\geq K\}}\rho(X_s^{0,y})J(\widehat{X}_s^{0,y})\alpha_n(s,X_s^{0,y})(f+(\partial_s+L)h_n)^-(s,X_s^{0,y})dsdy\\ \leq\ &E\int\rho(x)\left(\rho(x)^{-1}1_{\{|x|\geq K\}}\sup_{t\leq r\leq T}\rho(X_r^{0,x})J(\widehat{X}_r^{0,x})\right)K_{n,T}^{0,x}dx\\ \leq\ &\left(E\int\left(\rho(x)^{-1}1_{\{|x|\geq K\}}\sup_{t\leq r\leq T}\rho(X_r^{0,x})J(\widehat{X}_r^{0,x})\right)^2\rho(x)dx\right)^{1/2}\left(E\int\left(K_{n,T}^{0,x}\right)^2\rho(x)dx\right)^{1/2}\\ \leq\ &C\left(E\int\rho(x)^{-1}\left(1_{\{|x|\geq K\}}\sup_{t\leq r\leq T}\rho(X_r^{0,x})J(\widehat{X}_r^{0,x})\right)^2dx\right)^{1/2}\end{aligned}$$

the last inequality being a consequence of (21).

It is now sufficient to prove that

$$\int \rho(x)^{-1} E\left[\left(\sup_{0\leq r\leq T} \rho(X_r^{0,x}) J(\widehat{X}_r^{0,x})\right)^2\right] dx < \infty.$$

Note that

$$\begin{aligned} E\left[\left(\sup_{0\leq r\leq T} \rho(X_r^{0,x}) J(\widehat{X}_r^{0,x})\right)^2\right] &\leq (E \sup_{0\leq r\leq T} \left|\rho(X_r^{0,x})\right|^4)^{1/2} (E \sup_{0\leq r\leq T} \left|J(\widehat{X}_r^{0,x})\right|^4)^{1/2} \\ &\leq C(E \sup_{0\leq r\leq T} \left|\rho(X_r^{0,x})\right|^4)^{1/2}. \end{aligned}$$

So (24) permits to conclude.

Step 4. We are now able to construct the measure μ. Since the sequence $c_n = \nu_n([0,T]\times R^N), n \in N$, is bounded and $\nu_n, n \in N$ is tight, we may pass to a subsequence and get $\nu_n \to \nu$ where ν is a positive measure of total mass c. We define $\mu = \rho^{-1}\nu$ and so we have

$$\int_t^T \int \phi d\mu_n = \int_t^T \int \frac{\phi}{\rho} d\nu_n \to \int_t^T \int \frac{\phi}{\rho} d\nu = \int_t^T \int \phi d\mu$$

for every $\phi \in C([0,T]\times R^N)$ which has compact support with respect to $x \in R^N$.

Now, passing to the limit in the equation $\mathbf{PDE}(\mathbf{g}, \mathbf{f}, \mathbf{h}_n, \mathbf{q})$ one checks that (u, μ) verifies $\mathbf{PDE}(\mathbf{g}, \mathbf{f}, \mathbf{h}, \mathbf{q})$.

Let us now prove that μ satifies (19)b). We fix two continuous functions $\phi, \psi : [0,T]\times R^N \to R_+$ which have compact support in x and a continuous function with compact support $\theta : R^N \to R_+$ and write

$$\begin{aligned} & \int\int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) \psi(s,x)\theta(x) d\mu(s,x) \\ = & \lim_n \int\int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) \psi(s,x)\theta(x)\gamma_n(s,x) ds dx \\ = & \lim_n \int\int_t^T \phi(s,x)\psi(s, X_s^{t,x})\theta(X_s^{t,x})\gamma_n(s, X_s^{t,x}) ds dx \\ = & \lim_n \int\int_t^T \phi(s,x)\psi(s, X_s^{t,x})\theta(X_s^{t,x}) dK_{n,s}^{t,x} dx \\ = & \int\int_t^T \phi(s,x)\psi(s, X_s^{t,x})\theta(X_s^{t,x}) dK_s^{t,x} dx. \end{aligned}$$

We take $\theta = \theta_R$ to be the regularization of the indicator function of the ball of radius R and pass to the limit with $R \uparrow \infty$ in order to get (19) b).

We prove now that $\mu = 1_{\{u=h+q\}}\mu$. Using (18) iv) and v), $dK_s^{t,x} = 1_{\{u=h+q\}}(s, X_s^{t,x})dK_s^{t,x}$. Then (19) b) with $\psi = 1_{\{u=h+q\}}$ yields

$$\int\int_t^T \phi(s, \widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})1_{\{u=h+q\}}(s,x)d\mu(s,x) = \int\int_t^T \phi(s, \widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})d\mu(s,x).$$

Note that, $P(d\omega), a.s.$, the family of functions $A(\omega) = \{(s,x) \to \phi(\widehat{X}_s^{t,x}(\omega)) : \phi \in C_c^\infty\}$ is an algebra which separates the points (because $x \to \widehat{X}_s^{t,x}(\omega)$ is a bijection). Given a compact set K, $A(\omega)$ is dense in $C([0,T] \times K)$. It follows that $J(\widehat{X}_s^{t,x}(\omega))d\mu(dx,ds) = J(\widehat{X}_s^{t,x}(\omega))1_{(u=h+q)}(s,x)d\mu(dx,ds)$ for almost every ω. Since $J(\widehat{X}_s^{t,x}(\omega)) > 0$ for almost every ω, we get $d\mu(dx,ds) = 1_{(u=h+q)}(s,x)d\mu(dx,ds)$ and the proof of the existence is completed.

Uniqueness. Let $(\overline{u}, \overline{\mu})$ be any solution of $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ for which (19)b) holds true. We shall prove that $u = \overline{u}$ and $\mu = \overline{\mu}$. The proof follows the same line as in [B.M], so we just precise the main steps.

We fix $\varphi : R^N \to R$ a smooth function with compact support and denote $\varphi_t(s,x) = \varphi(\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})$. It is proved in [B.M], Proposition 3.3.2, that one may use $\varphi_t(s,x)$ as a test function in $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ provide one replaces $\partial_s\varphi_t(s,x)ds$ (which makes not sense) by a stochastic integral with respect to the semimartingale $(s,\omega) \to \varphi_t(s,x)$. One gets

$$\begin{aligned}
& \mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q}) \\
& \int\int_s^T \overline{u}d_r\varphi_t(r,x)dx - \int (g(x)\varphi_t(T,x) - \overline{u}(s,x)\varphi_t(s,x))dx \\
& - \int_s^T\int ((\sigma^*\nabla\overline{u})(\sigma^*\nabla\varphi_t) + \overline{u}\nabla(\widetilde{b}\varphi_t))dxdr \\
= & \int_s^T\int \varphi_t f(r,x,\overline{u},\sigma^*\nabla\overline{u})dxdr + \int_s^T\int \varphi_t(r,x)1_{(\overline{u}=h+q)}d\overline{\mu} \\
\forall t \leq & \; s \leq T, \forall\varphi \in C^{1,1}(R^N;R).
\end{aligned}$$

Moreover, by Corollary 3.3.1. in [B.M],

$$\begin{aligned}
\int\int_s^T \overline{u}d_r\varphi_t(r,x)dx = & \int_s^T\int (\sigma^*\nabla\overline{u})(\sigma^*\nabla\varphi_t) + \overline{u}\nabla(\widetilde{b}\varphi_t)dxdr \\
& + \int_s^T (\int (\sigma^*\nabla\overline{u})(r,x)_r\varphi_t(r,x)dx, dB_r).
\end{aligned}$$

We substitute this in $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ and get

$$\begin{aligned}
&\int \overline{u}(s,x)\varphi_t(s,x)dx \\
= \ &\int g(x)\varphi_t(T,x)dx + \int_s^T \int \varphi_t f(r,x,\overline{u},\sigma^*\nabla\overline{u})dxdr + \int_s^T \int \varphi_t(r,x)1_{(\overline{u}=h+q)}d\overline{\mu}(dx,dr) \\
&- \int_s^T (\int (\sigma^*\nabla\overline{u})(r,x)\varphi_t(r,x)dx, dB_r).
\end{aligned}$$

We make the change of variable $y = \widehat{X}_T^{t,x}$ and get

$$\int_{R^n} g(x)\varphi_t(T,x)dx = \int_{R^n} g(X_T^{t,y})\phi(y)dy$$

and we do the same thing for all the other terms, except for the one concerning $\overline{\mu}$ for which we use (19)b). We obtain

$$\begin{aligned}
&\int \overline{u}(s,X_s^{t,y})\varphi(y)dy \\
= \ &\int g(X_T^{t,y})\varphi(y)dy + \int_s^T \int \varphi(y) f(r,X_r^{t,y},\overline{u}(r,X_r^{t,y}),\sigma^*\nabla\overline{u}(r,X_r^{t,y}))dydr \\
&+ \int_s^T \int \varphi(y)1_{(\overline{u}=h+q)}(r,X_r^{t,y})d\overline{K}_r^{t,y}dy - \int_s^T \int (\sigma^*\nabla\overline{u})(r,X_r^{t,y})\varphi(y)dy, dB_r).
\end{aligned}$$

Since φ is arbitrary we have proved that for $\rho(y)dy$ almost every y, $(\overline{u}(s,X_s^{t,y}),\sigma^*\nabla\overline{u}(s,X_s^{t,y}),\overline{K}_s^{\prime,t,y})_{t\le s\le T}$ solves $E^{t,x}(g,f,h,q)$.

Here $\overline{K}_s^{\prime,t,y} = \int_t^s 1_{\{\overline{u}=h+q\}}(r,X_r^{t,y})d\overline{K}_r^{t,y}$.

Then, by the uniqueness property for the RBSDE we get $\overline{u}(s,X_s^{t,y}) = Y_s^{t,y} = u(s,X_s^{t,y})$ and $\overline{K}_s^{\prime,t,y} = K_s^{t,y}$. Taking $s=t$ we deduce that $\overline{u}(t,y) = u(t,y)$, $\rho(y)dya.s.$ and by using (19) b) we get

$$\int_s^T \int \varphi_t(r,x)1_{(\overline{u}=h+q)}(r,x)d\overline{\mu}(dx,dr) = \int_s^T \int \varphi_t(r,x)1_{(u=h+q)}(r,x)d\mu(dx,dr).$$

This allows to conclude (the same density argument as above) that $1_{(\overline{u}=h+q)}(r,x)d\overline{\mu}(dx,dr) = 1_{(u=h+q)}(r,x)d\mu(dx,dr)$.$\square$

We will now discuss the link between the above obstacle problem and the maximum principle. Let u^f be the weak solution, in variational sense, of the PDE (withought obstacle)

$$(\partial_t + L)u^f + f(t,x,u^f,\sigma^*\nabla u^f) = 0, \quad u_T = g.$$

Proposition 2 *(Maximum principle) i) Let $h \in C^{1,1}([0,T]\times R^d)$. There exists a positive measure $\mu_{f,h}$ such that if $g \ge h(T,\cdot)$ and*

$$F(t,x,y,z)dxdt \ge f(t,x,,y,z)dxdt + (f(t,x,h(t,x),\sigma^*\nabla h(t,x))dxdt + \mu_{f,h}(dxdt))^-, \forall y,z$$

then $u^F \geq h$.

ii) Suppose that f *does not depend on* z *and let* $h \in C([0,T]\times R^d)$. *There exists a positive measure* $\mu_{f,h}$ *such that if* $g \geq h(T,\cdot)$ *and*

$$F(t,x,y)dxdt \geq f(t,x,,y)dxdt + \left(f(t,x,h(t,x))dxdt + \mu_{f,h}(dxdt)\right)^{-}, \forall y$$

then $u^F \geq h$.

Proof. Let $(u,\mu_{f;h})$ be the solution constructed in the previous theorem for $PDE(g,f,h,0)$. Using a comparison argument $u^F \geq u \geq h$. □

Remark 15 *This result is somehow insatisfactory because we are not able to extend the enlightening relation* $f(t,x,h,\sigma^*\nabla h) + (\partial_t + L)h \geq 0 \Rightarrow \mu_{f,h} = 0$ *which holds true for a smooth* h. *We are also not able to prove that the measure* $\mu_{f,h}$ *produced by the obstacle problem is minimal.*

We turn now once again to the uniqueness property. The hypothesis (19) b) is very technical so we will prove uniqueness under more natural assumptions.

Theorem 5 *We assume that (15) , except for (15) ii), holds and* h *is continuous. Consider the weight* $\rho(x) = (1+|x|^2)^p, p \geq \frac{1}{2}\beta + 1$. *Assume also that* $\sigma\sigma^* \geq aI$ *for some strictly positive consant* a. *Then* $\mathbf{PDE(g,f,h,q)}$ *has a unique continuous solution.*

Proof. Let us check that the solution u which we have constructed in Theorem 7 is continuous. Assume first that $h \in C^{1,2}$. Using Theorem 6

$$u(t,x) = Eg(X_T^{t,x}) + \int_t^T E\Phi(s,X_s^{t,x})ds$$

where $\Phi(s,x) = F(s,x,\alpha(s,x),u(s,x),\sigma^*\nabla u(s,x))$. This may also be written as

$$u(t,x) = \int p_{T-t}(x,y)g(y)dy + \int_t^T \int p_{s-t}(x,y)\Phi(s,y)dyds$$

where $p_{s-t}(x,y)$ is the density of the law of $X_s^{t,x}$. Since $(t,x) \to p_{s-t}(x,y)$ is continuous it follows that u is continuous.

Assume now that h is just continuous and let $h_n, n \in N$, be a obtained by regularization by convolution. It is easy to see that for every $R > 0$,

$$\delta_{n,R} =: \sup_{|x|\leq R}\sup_{t\leq T} E \sup_{t\leq s\leq T}\left|h_n(s,X_s^{t,x}) - h(s,X_s^{t,x})\right|^2 \to 0$$

which, by (20) yields $\sup_{|x|\leq R}\sup_{t\leq T}|u_n(t,x) - u(t,x)| \leq C\delta_{n,R} \to 0$. So u is continuous.

Let us prove uniqueness. Suppose we know that uniqueness holds in the linear case, i.e. when f does not depend on y and z , and let us prove that it holds in the non linear case also. We are now in the general case and we denote by (u,μ) the solution constructed in Theorem 7 for $\mathbf{PDE(g,f,h,q)}$ and by $(\widetilde{u},\widetilde{\mu})$ another solution. Define $\widetilde{f}(t,x) =:$

$f(t,x,\widetilde{u}(t,x),\sigma^*\nabla\widetilde{u}(t,x))$. Clearly $(\widetilde{u},\widetilde{\mu})$ solves the linear equation $\mathbf{PDE}(\mathbf{g},\widetilde{\mathbf{f}},\mathbf{h},\mathbf{q})$ and so, by the uniqueness property for linear equations (which we assume true), it coincides with the solution (v,ν) constructed in Theorem 7 for this equation. In particular $\widetilde{\mu}$ satisfies the hypothesis (19) b), because ν does. Using the point iii) in Theorem 7 we conclude that $(u,\mu)=(\widetilde{u},\widetilde{\mu})$.

Let us now deal with the linear case. Let (u^1,μ^1) and (u^2,μ^2) be two solutions of $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ and let $u=u^1-u^2$ and $\mu=\mu^1-\mu^2$. We define $u_t^\varepsilon(x)=\varepsilon^{-1}\int_0^\varepsilon u(t+s,x)1_{\{t+s\leq T\}}ds$. It is easy to check that for every test function $\phi\in C^{1,1}([0,T]\times R^N)$ and every $0\leq t\leq T$

$$\int_t^T[(u_s^\varepsilon,\partial_s\phi_s)-e(u_s^\varepsilon,\phi_s)+(b\nabla u_s^\varepsilon,\phi_s)]ds+\int_t^T\int_{R^N}[\varepsilon^{-1}\int_0^\varepsilon\phi(t+r,x)dr]d\mu(s,x)=0$$

where $(.,.)$ is the scalar product in L^2 and $e(u,v)=:(\sigma^*\nabla u,\sigma^*\nabla v)$.

We write the above equation for the test function $u^\varepsilon\rho\pi_K$ where π_K is a regularization by convolution for the indicator function of $B_K=\{x:|x|\leq K\}$. ($u^\varepsilon\notin C^{1,1}([0,T]\times R^N)$ but, since $u\in H_\rho^1$, we may approximate u^ε by such functions). Note that

$$\int_t^T(u_s^\varepsilon,\partial_s u_s^\varepsilon\rho\pi_K)ds=\frac{1}{2}\int_t^T\partial_s\left\|u_s^\varepsilon\sqrt{\rho\pi_K}\right\|_{L^2}^2ds=-\frac{1}{2}\left\|u_t^\varepsilon\sqrt{\rho\pi_K}\right\|_{L^2}^2$$

so that the above equation becomes

$$\begin{aligned}&\frac{1}{2}\left\|u_t^\varepsilon\sqrt{\rho\pi_K}\right\|_{L^2}^2+\int_t^T e(u_s^\varepsilon,u_s^\varepsilon\rho\pi_K)ds\\ =\;&\int_t^T(b\nabla u_s^\varepsilon,u_s^\varepsilon\rho\pi_K)ds+\int_t^T\int_{R^N}[\varepsilon^{-1}\int_0^\varepsilon u_{s+r}^\varepsilon(x)dr]\rho\pi_K(x)d\mu(s,x).\end{aligned}$$

We write

$$e(u_s^\varepsilon,u_s^\varepsilon\rho\pi_K)\geq\left\|\sqrt{\rho\pi_K}\sigma^*\nabla u_s^\varepsilon\right\|_{L^2}^2-\delta_{\varepsilon,K}(s)$$

with $\delta_{\varepsilon,K}(s)=\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)(\sigma^*\nabla(\rho\pi_K))\right\|_{L^1}$. Recall that $\rho(x)=(1+|x|^2)^{-p}$ so that $|\nabla\rho|\leq C\rho$. It follows that

$$\begin{aligned}\delta_{\varepsilon,K}(s)&\leq C\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)\rho\pi_K)\right\|_{L^1}+C\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)\rho\nabla\pi_K)\right\|_{L^1}\\&\leq\frac{1}{2}\left\|\sqrt{\rho\pi_K}\sigma^*\nabla u_s^\varepsilon\right\|_{L^2}^2+C'\left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2+C\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)\rho\nabla\pi_K)\right\|_{L^1}.\end{aligned}$$

So

$$\frac{1}{2}\left\|\sqrt{\rho\pi_K}\sigma^*\nabla u_s^\varepsilon\right\|_{L^2}^2\leq e(u_s^\varepsilon,u_s^\varepsilon\rho\pi_K)+C'\left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2+C\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)\rho\nabla\pi_K)\right\|_{L^1}.$$

Moreover, for every $c>0$

$$|(b\nabla u_s^\varepsilon,u_s^\varepsilon\rho\pi_K)|\leq c\left\|\sqrt{\rho\pi_K}b\nabla u_s^\varepsilon\right\|_{L^2}^2+c^{-1}\left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2.$$

Since $\sigma\sigma^* \geq aI$, we may choose c such that $c\left\|\sqrt{\rho\pi_K} b\nabla u_s^\varepsilon\right\|_{L^2}^2 \leq \frac{1}{2}\left\|\sqrt{\rho\pi_K}\sigma^*\nabla u_s^\varepsilon\right\|_{L^2}^2$ so that

$$|(b\nabla u_s^\varepsilon, u_s^\varepsilon\rho\pi_K)| \leq e(u_s^\varepsilon, u_s^\varepsilon\rho\pi_K) + C'' \left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2 + \delta'_{\varepsilon,K}(s)$$

where $C'' = c^{-1} + C'$ and $\delta'_{\varepsilon,K}(s) = C\left\|u_s^\varepsilon(\sigma^*\nabla u_s^\varepsilon)\rho\nabla\pi_K)\right\|_{L^1}$.

Coming back to our equation we get

$$\begin{aligned}\frac{1}{2}\left\|u_t^\varepsilon\sqrt{\rho\pi_K}\right\|_{L^2}^2 \quad &\leq \quad C''\int_t^T \left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2 ds + \int_t^T \delta'_{\varepsilon,K}(s)ds \\ &\quad + \int_t^T\int_{R^N}[\varepsilon^{-1}\int_0^\varepsilon u_{s+r}^\varepsilon(x)dr]\rho\pi_K(x)d\mu(s,x).\end{aligned}$$

We pass to the limit with $\varepsilon \to 0$ in the above inequation. Clearly $\left\|\sqrt{\rho\pi_K}u_s^\varepsilon\right\|_{L^2}^2 \to \left\|\sqrt{\rho\pi_K}u_s\right\|_{L^2}^2$. Moreover

$$\begin{aligned}\int_t^T \delta'_{\varepsilon,K}(s)ds \quad &\leq \quad \int_t^T ds\int(|u_s^\varepsilon|^2 + |\sigma^*\nabla u_s^\varepsilon|^2)\rho\,|\nabla\pi_K|\,dx \\ &\leq \quad \int_t^T ds\int_{B_{K+1}^c}\varepsilon^{-1}\int_0^\varepsilon(|u_{s+r}|^2 + |\sigma^*\nabla u_{s+r}|^2)dr\rho dx \\ &\leq \quad \int_0^T ds\int_{B_{K+1}^c}(|u_s|^2 + |\sigma^*\nabla u_s|^2)\rho dx =: \delta_K.\end{aligned}$$

Finally, since u is continuous

$$\int_t^T\int_{R^N}[\varepsilon^{-1}\int_0^\varepsilon u_{r+s}^\varepsilon(x)dr]\rho\pi_K(x)d\mu(s,x) \to \int_t^T\int_{R^N} u_s\rho\pi_K(x)d\mu(s,x).$$

Since μ^i is concentrated on $\{u^i = h+q\}$, $\rho\pi_K u d\mu = \rho\pi_K[(h+q-u^2)d\mu^1 + (h+q-u^1)d\mu^2] \leq 0$. So we get

$$\frac{1}{2}\left\|u_t\sqrt{\rho\pi_K}\right\|_{L^2}^2 \leq C''\int_t^T \left\|\sqrt{\rho\pi_K}u_s\right\|_{L^2}^2 ds + \delta_K.$$

Gronwall's lemma yields $\left\|u_t\sqrt{\rho\pi_K}\right\|_{L^2}^2 \leq C\delta_K$. Since $u \in H_\rho^1$, $\delta_K \to 0$ as $K \to \infty$, so, by passing to the limit we get $\left\|u_t\sqrt{\rho}\right\|_{L^2} = 0$.□

Remark 16 *One may wonder wether the analytical proof used when f does not depend on y, z would prove uniqueness in the general case as well. The reasoning breaks down when one evaluates*

$$\begin{aligned}&\varepsilon^{-1}\left|\int_0^\varepsilon[f(t+s,x,u_{t+s}^1,\sigma^*\nabla u_{t+s}^1) - f(t+s,x,u_{t+s}^2,\sigma^*\nabla u_{t+s}^2)]ds\right| \\ \leq \quad & C\varepsilon^{-1}\int_0^\varepsilon(\left|u_{t+s}^1 - u_{t+s}^2\right| + \left|\sigma^*\nabla(u_{t+s}^1 - u_{t+s}^2)\right|)ds.\end{aligned}$$

Since $s \to |u^1_s - u^2_s|$ is continuous, $\varepsilon^{-1}\int_0^\varepsilon |u^1_{t+s} - u^2_{t+s}|\,ds \to |u^1_t - u^2_t|$ so we may handle this term. But $s \to |\sigma^* \nabla(u^1_{t+s} - u^2_{t+s})|$ is not continuous any more so we may not pass to the limit. On the other hand $\varepsilon^{-1}\int_0^\varepsilon |\sigma^*\nabla(u^1_{t+s} - u^2_{t+s})|\,ds$ is not dominated by $|\varepsilon^{-1}\int_0^\varepsilon \sigma^*\nabla(u^1_{t+s} - u^2_{t+s})ds|$ so that the cancelling procedure based on the energy does not operate. It seems that the more general framework in which the above analitical reasoning works is $f(t,x,y,z) = f_1(t,x,y) + f_2(x)z$.

3.3 Variational inequalities

In [Be.L] A. Bensoussan and J.L. Lions associate to the obstacle problem that we discuss here the following variational inequality:

$$
\begin{aligned}
(26)\quad i)(\partial_t + L)u + f &\leq 0,\\
ii)((\partial_t + L)u + f)(u - h - q) &= 0\\
iii)u &\geq h + q\\
iv)u_T &= g.
\end{aligned}
$$

Since generally this problem does not have a classical solution, they give the following weak formulation. A solution of the variational inequality (26) is a function $u \in L^2([0,T], H^1_\rho)$ such that $u \geq h + q, a.s.$ and such that

$$
(27)\quad \int_0^T (\partial_t\phi_t, u_t - \phi_t)dt - \int_0^T A(u_t, u_t - \phi_t)dt + \int_0^T (f_t, u_t - \phi_t)dt + \frac{1}{2}|\phi_T - g|^2 \geq 0
$$

for every $\phi \in C^{1,1}$ such that $\phi \geq h + q, a.s.$

Here $(.,.)$ designees the scalar product in $L^2(dx)$ and $A(u,\phi) = (\sigma^*\nabla u)(\sigma^*\nabla\phi) + u\nabla(\widetilde{b}\phi)$.

In [Be.L], Theorem 2.6 (see also F. Mignot and J.P.Puel [M.P]) they prove that the above variational inequality has a solution. This solution is generally not unique but they prove that there exists a unique minimal solution , *i.e.* a solution u such that, for any other solution v, one has $u \leq v$.

Remark 17 *In fact they consider an upper barrier, so they look for a maximal solution. As for the hypotheses, they are of the same kind as here with the following differences: the assumption on the obstacle is slightly weaker but they assume uniform ellipticity.*

The relation between the $\mathbf{PDE(g,f,h,q)}$ and the variational inequality is given in the following theorem:

Theorem 6 *Let (u,μ) be the solution of the $\mathbf{PDE(g,f,h,q)}$ constructed in the previous section. Then u solves (27). If in addition the obstacle $h+q$ is a continuous function of (t,x) and if $\sigma\sigma^* \geq cI$, then u is the minimal solution.*

Proof. We need the following equality (which we prove at the end):

$$\int_0^T A(u_t,u_t)dt = \frac{1}{2}(|g|^2 - |u_0|^2) + \int_0^T (f_t,u_t)dt + \int_0^T \int u(t,x)d\mu(t,x) \tag{28}$$

where $f_t = f(t,x,u(t,x),\sigma^*\nabla u(t,x))$.

We assume for a moment that (28) holds and we prove (27). For a test function $\phi \in C^{1,1}$ we have

$$\int_0^T (\partial_t\phi_t,\phi_t)dt = \frac{1}{2}(|\phi_T|^2 - |\phi_0|^2) \tag{29}$$

and so, using the $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$ we get

$$\begin{aligned} & \int_0^T (\partial_t\phi_t,u_t-\phi_t)dt - \int_0^T A(u_t,u_t-\phi_t)dt + \frac{1}{2}(|\phi_T|^2 - |\phi_0|^2) \\ & +\frac{1}{2}(|g|^2 - |u_0|^2) + \int_0^T (f_t,u_t)dt + \int_0^T \int u(t,x)d\mu(t,x) \\ = \quad & (\phi_T,g) - (\phi_0,u_0) + \int_0^T (f_t,\phi_t)dt + \int_0^T \int \phi(t,x)d\mu(t,x). \end{aligned}$$

Since

$$\begin{aligned} & \frac{1}{2}(|\phi_T|^2 - |\phi_0|^2) + \frac{1}{2}(|g|^2 - |u_0|^2) - (\phi_T,g) + (\phi_0,u_0) \\ = \quad & \frac{1}{2}|\phi_T - g|^2 - \frac{1}{2}|\phi_0 - u_0|^2 \end{aligned}$$

one gets

$$\begin{aligned} & \int_0^T (\partial_t\phi_t,u_t-\phi_t)dt - \int_0^T A(u_t,u_t-\phi_t)dt + \int_0^T (f_t,u_t-\phi_t)dt + \frac{1}{2}|\phi_T - g|^2 \\ = \quad & \frac{1}{2}|\phi_0 - u_0|^2 + \int_0^T \int (\phi - u)(t,x)d\mu(t,x). \end{aligned}$$

Recall that μ is a positive measure which is concentrated on the set $\{u = h+q\}$, so, if $\phi \geq h+q$, (which is the case for our test functions) the second member of the above inequality is positive and (27) is proved.

Let us now prove (28). Suppose for a moment that $u \in C^{1,1}$ so that we may take it as a test function in $PDE(g,f,h,q)$. Then using the equality (29) for u we get (28). But generally $u \notin C^{1,1}$ so we have to use an approximation procedure: we denote by $\overline{\gamma}_n$ the regulariztion by convolution of γ_n ,which is defined in the proof of Thorem 7, (resp.f_n the regularisation by convolution of f) and by $\overline{u}_n$ the solution of the PDE $(\partial_t + L)\overline{u}_n + f_n + \overline{\gamma}_n = 0$. Then $\overline{u}_n \in C^{1,2}$ so we have

$$\int_0^T A(\overline{u}_n(t),\overline{u}_n(t))dt = \frac{1}{2}(|g|^2 - |\overline{u}_n(0)|^2) + \int_0^T (f_n(t) + \overline{\gamma}_n(t),\overline{u}_n(t))dt$$

The same arguments as in the proof of Theorem 7 show that $\overline{u}_n \to u$ in H^1_ρ so we pass to the limit and get (28) for u.

Assume now that $h+q$ is continuous and $\sigma\sigma^* \geq cI$ and let us prove that, if v solves (27), then $v \geq u$. Let u_n be the solution of the penalized PDE, that is $(\partial_t+L)u_n+f(t,x,u,\sigma^*\nabla u)+n(u_n-h-q)^- = 0$, $u_n(T,x) = g(x)$. It is proved in A Bensoussan, J.L. Lions [Be.L] Ch IV, Th 1.6, that if v solves (27), then $v \geq u_n$. So the only thing we have to prove is that $u_n \to u$.

Let $Y^{t,x}_{n,s} = u_n(s, X^{t,x}_s)$, $Z^{t,x}_{n,s} = \sigma^*\nabla u_n(s, X^{t,x}_s)$. It is proved in [B.M] that $(Y^{t,x}_{n,s}, Z^{t,x}_{n,s})_{t\leq s\leq T}$ solves the BSDE

$$Y^{t,x}_{n,s} = g(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_{n,r}, Z^{t,x}_{n,r}) + n(Y^{t,x}_{n,r} - (h+q)(r, X^{t,x}_r))^- dr - \int_s^T (Z^{t,x}_{n,r}, dB_r)$$

and it is proved in [El-K.K.P.P.Q] that, for each fixed (t,x), $u_n(t,x) = Y^{t,x}_{n,t} \to Y^{t,x}_t = u(t,x)$ (Here the continuity of the obstacle is needed). The proof is completed.$\square$

Remark 18 *The equation (29) (which is equivalent to the* $\mathbf{PDE}(\mathbf{g},\mathbf{f},\mathbf{h},\mathbf{q})$*) represents a more precise formulation of the variational inequality (27). Actually the fact that (27) is just an inequality allows to avoid to deal with the measure* μ*. The difficulty in constructing* μ *consists in the fact that* μ *appears as the limit of the sequence* $\mu_n(ds,dx) = \phi_n(f+(\partial_s+L)h_n)^-(s,x)dsdx$*. Because of the negative part* $(...)^-$ *one cannot use the integration by parts and so one can not obtain a variational formulation for the equation. So, a clever way to avoid this difficulty is to weaken the equality up to an inequality. Using the evalutions given by the RBSDE (the relation between* μ_n *and the corresponding increasing process* K_n*), we solve this difficulty and produce the measure* μ*. This may be viewed as a regularity result. Anyway it is to be stressed that we assume more regularity on the obstacle (see (17)). If* h *is just an element of* L^2 *we can say nothing.*

References

[B.C.F] V.Bally,M.C.Caballero,B.Fernandez: Reflected BSDE's,PDE's and Variational Inequalities, Prperint of Laboratoire de Probabilites, Universite Paris VI, PMA-492, 1999

[B.M] V Bally, A. Matoussi: Stochastic PDE's and Doubly Stochastic Backward Differentiable Equations. J. of Theoretical Probability, ...

[B.P.S] V. Bally, E. Pardoux, L. Stoica: BSDE's associated to semilinear PDE's with measurable coefficients. In preparation.

[Ba.L] G. Barles, E. Lesigne: SDE,BSDE and PDE. In "Bakward Stochastic Differential Equations", N. El Karoui and L. Mazliak, editors, Pitman Research Notes in Math. Series, 364, Longman, 1997.

[Be.L] A. Bensoussan, J. L. Lions: Applications des Inéquations variationelles en control stochastique. Dunot, Paris, 1978.

[C.K] J. Cvitanic, I. Karatzas: Backward Stochastic Differential Equations with reflection and Dynkin games. The Annals of Proba. 1996, Vol 24, no.4, 2024-2056.

[D.S] N. Dunford, J.T. Schwarz: Linear operators I: General theory. Whily, New York, 1963.

[E.T] I. Ekeland,R. Temam. Convex analysis and variational problems. North Holland, Amsterdam, 1976.

[El-K.P.Q] N. EL Karoui, S. Peng, M.C. Quenez: Backward stochastic differential equations in finance. Math. Finance. 1994.

[El-K.K.P.P.Q] N. El Karoui, C. Kapoudjan, E. Pardoux, S. Peng, M.C. Quenez: Reflected solutions of Backward SDE's, and related obstacle problems for PDE's. The Annals of Probability,1997, Vol. 25,No 2, 702-737.

[H.L.M] S. Hamaden, J.P.Lepeltier, A. Matoussi : Double barrier backward SDEs with continuous coefficient. In "Bakward Stochastic Differential Equations", N. El Karoui and L. Mazliak, editors, Pitman Research Notes in Math. Series, 364, Longman, 1997,177-193.

[H.L,1] S. Hamadene, LJ.P. Lepeltier: Backward Equations,Stochastic Control and Zero-Sum Stochastic Differential Games. Stoch. and Stoch. Repports, Vol 54, p. 221-231, 1995.

[H.L,2] S. Hamaden, J.P.Lepeltier: Reflected Backward SDE's, mixed control problems and game problems.

[K.1] H. Kunita: Stochastic flow acting on Schwartz distributions. Journal of theoretical Probability, Vol 7, 247-278,1994.

[K.2] H. Kunita: Generalized Solutions of a Stochastic Partial differential Equation. Journal of theoretical Probability. Vol 7, 2, 279-308,1994.

[I.W] N.Ikeda and S.Watanabe: Stochastic Differential Equations and Diffusion Processes. North-Holland, 1989.

[M.P] F. Mignot, J.P. Puel: Solutions maximum de certaines inéquations d'evolution paraboliques et inéquations quasi-variationnelles paraboliques,C.R.A.S. 280, série A, 1975, p 259.

[P.P.1] E. Pardoux, S. Peng: Adapted solution of a backward stochastic differential equation. Systems Control Letters, 14, 55-61, 1990.

[P.P.2] E. Pardoux, S. Peng: Backward SDE's and quasilinear PDE's. In Stochastic Partial Differential Equations and their Applications, B.L. Rozovskii and R.B. Sowers, editors. Lecture Notes in Control and Inform. Sci. 176, Springer, 1992,Berlin.

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